Financial Frictions in Dynamic Economics

Class Notes from Econ 589

Costas Azariadis, Fall 2008

Lecture 1: Introduction to the course

General topic: Implications of financial frictions and limited factor mobility in economies with sectoral and idiosyncratic shocks

a) Frictions: -- Collateral and reputational borrowing under endogenous debt limits. Complete markets (no equilibrium default) and incomplete markets (default in equilibrium)

-- Secured or collateral borrowing (borrow up to the present value of what your creditors can force you to repay).

-- Unsecured or reputational borrowing (borrow up to the point where you are indifferent between solvency and default. Default destroys reputation and inhibits future borrowing).

b) Implications to be studied:

-- consumption smoothing in exchange economies

-- business cycles in economies without aggregate or economy-side shocks (amplification, persistence and other dynamic properties of detrended GDP)

-- development and growth (poverty traps and large growth fluctuations in emerging economies)

-- distribution of wealth (financial wealth Gini index higher than income Gini index)

-- demand for money and the proper role for monetary policy (social insurance, equilibrium selection, etc)

c) Key issue: -- Understanding economic volatility. Can small and infrequent shocks have large and persistent consequences? How should policy react to these consequences?

Contents of the class

1. The logic of debt limits: Both unsecured and secured borrowing restrict loans

below the borrower's present value of future income.

Debt limits restrict arbitrage

⇨ Resources do not necessarily flow to the consumers or firms who value resources the most

⇨ Limits on capital mobility

⇨ Misallocation of resources and productive inputs like capital and labor

2. Debt limits in dynamic general equilibrium (DGE):

a) need for DGE:

-- simplicity

-- unified framework for all of macroeconomics

b) need for frictions:

-- poor match of DGE economies without frictions (eg., the RBC model) with data from business cycle dynamics, international development patterns, and asset returns

3. RBC vs the data:

Typical RBC model

$$V_0 = E_0\{\sum_{t=0}^{\infty} \beta^t [\log C_t + \gamma \log(1-L_t)]\} \qquad 0 < \beta < 1, \quad \gamma > 0$$

$= \text{representative household payoff at } t = 0.$

Technology: $F(K_t, L_t) = Z_t \; K_t^{\alpha} \; L_{1-t}^{1-\alpha}$,

where $\log(Z_t) = \rho \log(Z_{t-1}) + \varepsilon_t$ = Solow residual,

$\rho \in (0,1), \quad \varepsilon_t \sim i.i.d., \quad 0 < \alpha < 1.$

Equilibrium: Dynamic properties of output are exactly those of the Solow residual. Output is as persistent as Z_t, shocks do not build up or amplify. Convergence to steady state, following a shock, is rapid.

Initial Reading List

Eaton and Gersovitz, RES 1981 (first attempt to describe borrowing with limited enforcement)

Lilien, JPE 1982 (first empirical attempt to connect business cycles with sectoral shocks)

Bulow and Rogoff, AER 1989 (reputational borrowing with weak default penalties)

Hsieh and Klenow, working paper 2007 (international evidence on misallocation)

Kehoe and Levine, RES 1993 (reputational borrowing with strong default penalties)

Kocherlakota, RES 1996 (elaboration of Kehoe and Levine)

Kiyotaki and Moore, JPE 1997 (collateral borrowing)
Kiyataki, JER 1998 (business cycle implications of collateral borrowing)
Alvarez and Jermann, Econometrica 2000 (endogenous debt limits for reputational borrowing with strong default penalties)
Hellwig and Lorenzoni, working paper, 2007 (extends Bulow-Rogoff model)

Financial Frictions in Dynamic Economics

Class Notes from Econ 589

Costas Azariadis, Fall 2008

Lecture 2: The Kiyotaki-Moore model of collateral borrowing (JPE 1997)

1. Introduction

Aim: Reduce role of TFP shocks in explaining business cycles

Goals are: 1) small i.i.d. TFP shocks to have large and persistent impact on output
2) hump-shaped impulse response to a one-time TFP shock

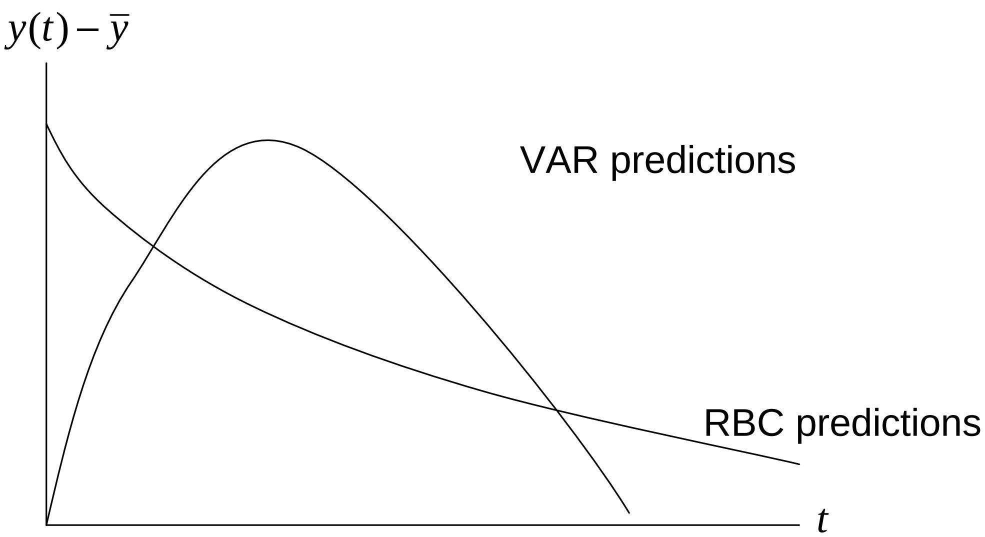


Figure 1

Mechanism: Positive TFP shock changes value of collateral assets
⇨ relaxes debt limits
⇨ increases borrowing
⇨ increases investment
⇨ increases output

Outcome: Persistence is easy to obtain even if shocks are i.i.d.
Amplification is harder with one-time shock, easier with persistent shock.

Hump-shaped impulse response is hardest.

2. Literature: Kiyotaki 1998, Japanese Economic Review
Kocherlakota 2000, Federal Reserve Bank of Minneapolis Quarterly Review, and 2008 Working Paper
Pintus and Wen, FRB St. Louis, working paper #2008-014B

3. The Simplest Model: An economy without physical capital

There are two types of infinitely-lived agents, $i = 1, 2$.

a) A patient consumer/lender with risk-neutral utility function

$$V_1 = \sum_{t=0}^{\infty} \beta_1^t (C_t^1 + b\, L_t^1)$$

where C_t^1 =consumption of goods, and L_t^1 =use of land or housing, $0 < \beta_1 < 1$, $b > 0$ represents the marginal utility of land. The household has no endowment of goods or land. Land is in unit net supply, not reproducible, may be purchased but cannot be rented. There is no rental market. P_t is the price of land in terms of the consumption good. The household's budget constraint is

$$C_t^1 + P_{t+1}(L_{t+1}^1 - L_t^1) + B_{t+1} \le R_t\ B_t,$$

where B_t =loans made at $t-1$ and maturing at t with yield $R_t = 1 + r_t$.

b) An impatient consumer/borrower/producer with a risk-averse utility function

$$V_2 = \sum_{t=0}^{\infty} \beta_2^t \log C_t$$

where $0 < \beta_2 < \beta_1 < 1$.

The household has the following constraints:

i) Technology: $Y_t = A_t\, L_t^{\gamma}$, where A_t is exogenous, $\gamma \in (0, 1]$, L_t is land input

ii) The debt limit on borrowing: $R_t\ B_t \le P_t\ L_t$, which states that land is good collateral but the consumption good is not.

iii) Budget constraint: $C_t + P_{t+1}(L_{t+1} - L_t) + R_t\ B_t \le B_{t+1} + Y_t$.

iv) There is no capital or labor input.

c) Equilibrium

-- budget constraints hold at equality
-- f.o.c's for all agents apply
-- markets for the consumption good and land clear:

$$C_t^1 + C_t = Y_t,$$

$$L_t^1 + L_t = 1, \quad \forall t$$

d) F.O.C's for the lender

$$1 = \beta_1 \ R_{t+1}$$

$$\Rightarrow \quad R_{t+1} = 1/\beta_1 = 1+\rho, \quad \forall t$$

where ρ =lender's rate of time preference.

The no-arbitrage requirement means that the rates of return on land and on loans must be equal at any interior equilibrium, that is,

$$\Rightarrow \quad \frac{1}{\beta_1} = \frac{P_{t+1}+b}{P_t}$$

where b is the consumer's implied "rental" or dividend from land. This equation has a unique perfect-foresight solution

$$\Rightarrow \quad P_t = P^* = \frac{b}{\rho}, \quad \forall t .$$

(which is exactly the present value of the constant dividend stream.)

In principle, it may be possible to have a corner equilibrium at which

$$\frac{1}{\beta_1} > \frac{P_{t+1}+b}{P_t}.$$

If this happens, the lender's demand for land drops to zero, and all land is owned by the borrower.

e) F.O.C.'s for borrower

$$\max \ V_2 = \sum_{t=0}^{\infty} \beta_2^t \ \log C_t$$

$$s.t. \quad C_t = B_{t+1} - R\ B_t - P_{t+1}(L_{t+1} - L_t) + Y_t$$

$$R\ B_t \le P\ L_t$$

where $Y_t = A_t\ L_t^\gamma$, and the second constraint is with a multiplier $\beta_2^t\ \lambda_t \ge 0$, where $\lambda_t \ge 0$, with $\lambda_t = 0$ when $R\ B_t < P\ L_t$.

Here we write down the Lagrangean:

$$\Lambda = \sum \beta_2^t \left\{ \log C_t + \lambda_t (P\ L_t - R\ B_t) \right\}$$

Compute the f.o.c. with respect to lending:

$$0 = \frac{\partial \Lambda}{\partial B_t} = -\beta_2^t \lambda_t R + \beta_2^t \frac{1}{C_t}\frac{\partial C_t}{\partial B_t} + \beta_2^{t-1}\frac{1}{C_t}\frac{\partial C_{t-1}}{\partial B_t}$$

$$\Rightarrow \quad 0 = -\beta_2 R \lambda_t + \frac{\beta_2}{C_t}(-R) + \frac{1}{C_{t-1}}$$

$$\Rightarrow \quad \frac{1}{C_{t-1}} = \frac{\beta_2 R}{C_t} + \beta_2 R\lambda_t = \frac{\beta_2}{\beta_1}(\lambda_t + \frac{1}{C_t}) \tag{1a}$$

Compute the f.o.c. with respect to land:

$$0 = \frac{\partial \Lambda}{\partial L_t} = \beta_2^t \lambda_t P + \beta_2^t \frac{1}{C_t}\frac{\partial C_t}{\partial L_t} + \beta_2^{t-1}\frac{1}{C_t}\frac{\partial C_{t-1}}{\partial L_t}$$

$$\Rightarrow \quad 0 = \beta_2 \lambda_t P + \frac{\beta_2}{C_t}(P + MPL_t) - \frac{1}{C_{t-1}} P$$

where MPL_t =marginal product of land at t. Then

$$0 = \beta_2 \lambda_t P + \frac{\beta_2}{C_t}(P + MP_L) - \frac{1}{C_{t-1}} P$$

$$\Rightarrow \quad \frac{1}{C_{t-1}} = \beta_2\left[\lambda_t + \left(\frac{P + MPL_t}{P}\right)\frac{1}{C_t}\right]. \tag{1b}$$

Notice that $\left(\frac{P^* + MPL_t}{P^*}\right)$ is the ROR on land as defined by the producer/borrower.

Kiyotaki-Moore show that binding debt constraints contribute to the persistence and size of economic fluctuations. In particular:

1) If TFP shocks are i.i.d., then GDP is also i.i.d. when the debt constraint is slack, but it is autoregressive, that is AR(1), when the debt constraint binds.
2) Output response may be bigger than TFP shock (amplification effect)

Equating the right-hand side of (1a) and (1b), we obtain

$$\rho\, \lambda_t = \left(\frac{MPL_t}{P} - \rho\right)\frac{1}{C_t} \tag{2a}$$

which is equal to 0 if $MPL_t = b$, and greater than 0 if $MPL_t > b$.

Eliminating λ_t from (1a) and (1b) leads to

$$\frac{C_t}{C_{t-1}} = \frac{\beta_2}{\beta_1}\frac{MPL_t}{P^*}$$

So we conclude that, if $\lambda_t > 0$, the impatient household's consumption falls at a constant rate, i.e. $C_t / C_{t-1} = \beta_2 / \beta_1 < 1$.

4. Equilibrium without binding debt limits: Output as persistent as shocks

Suppose $\lambda_t = 0, \forall t$. Then

$$\frac{C_t}{C_{t-1}} = \frac{\beta_2}{\beta_1} < 1$$

$$MPL_t = b = \frac{\gamma\, A_t}{L_t^{1-\gamma}}$$

$$Y_t = A_t\ L_t^{\gamma} = A_t \left(\frac{\gamma A_t}{b}\right)^{\gamma/(1-\gamma)} = k\ A_t^{\frac{1}{1-\gamma}}$$, for some constant $k > 0$.

Therefore, we can rewrite output as

$$\log Y_t = \log k + \frac{1}{\gamma} \log A_t$$

$$C_t^1 + C_t = Y_t$$

Then, if the TFP shock is i.i.d., so is Y_t .

Question: Is $R\ B_t < P\ L_t$ in this equilibrium?

Answer: Compute B_t from budget constraint of either agent. Then the debt constraint is slack if and only if β_1 / β_2 is not too large.

5. Interior equilibrium with binding debt limits: Output more persistent than shocks

Suppose next that β_1 / β_2 is sufficiently large. Assume now that $\lambda_t > 0$, $\forall t$. Then $R\ B_t = P\ L_t$, and the borrower's budget constraint becomes

$$C_t = Y_t + B_{t+1} - R\ B_t - P(L_{t+1} - L_t)$$

$$\Rightarrow \quad C_t = Y_t + \left(\frac{1}{R} - 1\right) P\ L_{t+1}$$

$$= Y_t - \beta_1\ b\ L_{t+1}$$

Equilibrium satisfies a budget constraint and a consumption-Euler equation for the borrower:

$$\begin{cases} Y_t = C_t + \beta_1\ b\ L_{t+1} \\ \dfrac{C_t}{C_{t-1}} = \dfrac{\beta_2}{\beta_1} \dfrac{MPL_t}{b} \end{cases}$$

To solve for the unknown sequence (C_t, L_t) , we conjecture a solution of the form

$$C_t = (1-m)\ A_t\ L_t^{\gamma} \quad (3a)$$

$$\beta_2\ b\ L_{t+1} = m\ A_t\ L_t^{\gamma} \quad (3b)$$

where $m \in (0,1)$ is a parameter to be determined. This solution satisfies the budget constraint for all $m \in (0,1)$. It also satisfies the consumption growth equation if

$$\frac{A_t\ L_t^{\gamma}}{A_{t-1}\ L_{t-1}^{\gamma}} = \frac{\beta_2}{\beta_1} \frac{\gamma A_t}{L_t^{1-\gamma}} \frac{1}{b}$$

$$\Rightarrow \quad L_t = A_{t-1} \frac{\beta_2}{\beta_1} \frac{\gamma}{b} L_{t-1}^{\gamma}$$

$$\Rightarrow \quad \frac{m\, A_{t-1}\, L_{t-1}^{\gamma}}{\beta_1\, b} = A_{t-1} \frac{\beta_2}{\beta_1} \frac{\gamma}{b} L_{t-1}^{\gamma}$$

$$\Rightarrow \quad m = \gamma\, \beta_2$$

Then from the technology and equation (3b), we obtain

$$L_{t+1} = \left(\frac{Y_{t+1}}{A_{t+1}} \right)^{1/\gamma} = \left(\frac{m}{b\, \beta_2} \right) Y_t = \left(\frac{\gamma}{b} \right) Y_t$$

$$\Rightarrow \quad Y_{t+1} \text{ is proportional to } A_{t+1}\, Y_t^{\gamma}.$$

Dynamics of constrained output is then

$$\log Y_{t+1} = \phi + \gamma \log Y_t + \log A_{t+1}$$

for some constant ϕ. Lagged output enters as an autoregressive term due to a binding collateral constraint.

Conclusion. Output is serially correlated even if TFP shocks are not.

Question: What about amplification and a hump-shaped impulse response?

Answer: -- K-M model: use non-convex technology

-- Pintus and Wen, "Boom-debt cycles," Federal Reserve Bank of St. Louis WP#7008-014B: use habit formation

-- An alternative is to use an enduring shock, i.e., a unit shock to output at $t = T$ and $T+1$ that lasts two periods, instead of one. The impulse response will have a hump, that peaks in the second period, $t = T+1$, and declines monotonically thereafter.

Problem 2.1: Describe equilibria with binding collateral constraints in a stochastic version of the simplest Kiyotaki-Moore model with stochastic technology in which $\log A_t$ is AR(1) process, i.e. $\log A_t = \rho \log A_{t-1} + \varepsilon_t$, for $\rho \in (0,1)$ and $\varepsilon_t \sim i.i.d.$, representing an aggregate technology shock. Show that $\log Y_t$ is AR(2) and find the response of $\log Y_t$ to a unit shock in the i.i.d. term ε_t. What are the eigenvalues of the GDP process?

Problem 2.2: Set $A_t = 1$ for all t in the deterministic version of the Kiyotaki-Moore model.

(a) Are there corner equilibria in which all land is held by the borrower? If so, describe the equilibrium price of land. In particular, is $p_t > b/\rho$?

(b) Are there corner equilibria in which all land is owned by the lender?

Financial Frictions in Dynamic Economics

Class Notes from Econ 589

Costas Azariadis, Fall 2008

Lecture 3: Exchange economies with unsecured borrowing

1. Ideas

(i) Unsecured or reputational borrowing only, without any commitment to repay. Default destroys reputation.

(ii) Perfect information but limited enforcement
Contracts are enforced by the threat of exclusion from asset markets:
- ⊙ Kehoe-Levine: Perpetual two-sided exclusion (no borrowing or lending after default)
- ⊙ Bulow-Rogoff: Perpetual one-sided exclusion (can lend out after default but cannot borrow ever)

(iii) Complete markets leave no social role for default in equilibrium (Default deterred for all borrowers in all possible histories of events by endogenous debt limits on each borrower)

(iv) Binding debt limits reduce capital mobility and inhibit equalization of MRS's among consumers and MRK's among firms
$\Rightarrow$ misallocation of resources (Hsieh and Klenow, 2007)

(v) Idiosyncratic shocks are one order of magnitude larger than common or aggregate shocks (Davis and Haltiwanger)

2. Goals

Study misallocation in exchange economies and in production economies. What can monetary or fiscal policy contribute to improve capital mobility and lessen misallocation?

Explore reasons why, in economies with complete markets and limited enforcement, the 1st welfare theorem fails and the 2nd welfare theorem holds.

Demonstrate that, in economies with periodic sectoral or idiosyncratic shocks, aggregate TFP and growth rate may fluctuate, as shown below, even though the PPF is stable. More generally, sectoral shocks can amplify or propagate common or aggregate shocks.

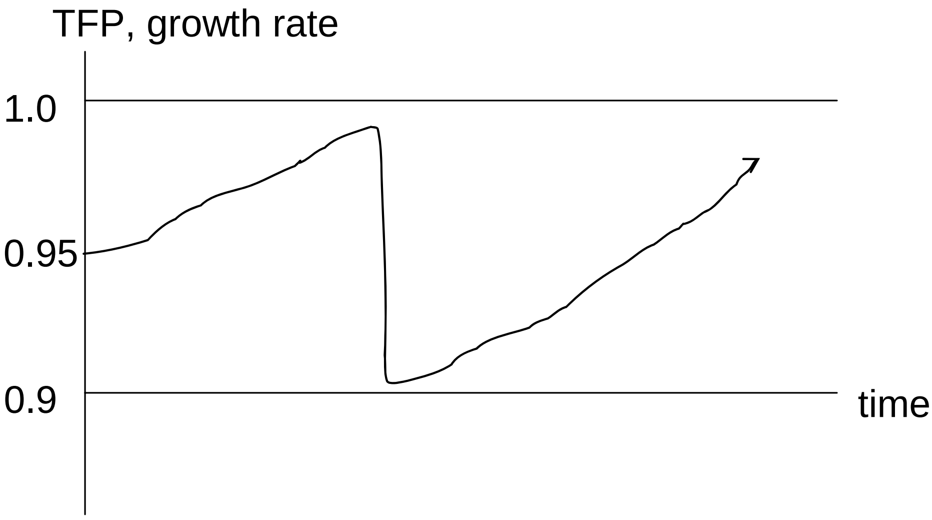


Figure 2

3. Road Map for Remaining Lectures

a) The Arrow-Debreu model economy as a benchmark. (this lecture)
Complete markets with perfect enforcement.

b) Complete markets with limited enforcement. Unsecured borrowing in deterministic exchange economies (lecture 4) with two-sided and one-sided exclusion, and in stochastic exchange economies (lecture 5). Constrained optima. Dynamic complementarities and multiple Pareto-ranked equilibria.

c) Collateral and reputational borrowing in production economies (lectures 6 through 8). Capital misallocation and growth fluctuations. The aggregate Solow residual.

d) The Aiyagari economy with incomplete markets and non-contingent claims (lectures 9 and 10).
Consumption smoothing with and without default.

e) Monetary and fiscal policy in economies with financial frictions (lectures 10 and 11). Equilibrium selection, social insurance, and collateral provision.

f) Money and bankruptcy as substitutes (lecture 12). What feedbacks are there? How should monetary policy respond to the state of financial markets?

4. Setting

We describe a pure exchange economy with constant total income in which assets are claims on future income and exist in zero net supply. We focus on the distribution of consumption over households first, and later on the distribution of production over firms, all of which are assumed to suffer from idiosyncratic or sectoral shocks. Throughout these notes we suppose that:

Assumption 3.1: Total income and consumption possibilities are constant in each exchange economy.

Assumption 3.2: The aggregate production possibility frontier is stable in each production economy.

We analyze economies with sectoral shocks as well as economies with purely idiosyncratic shocks. All households have the same expected income in the long run. States of nature are represented by $s_t \in \{L, H\}$, that is, by a binary Markov process with transition probabilities

$$\pi(s, s') = prob\{s_{t+1} = s' | s_t = s\},$$

where $\pi_{ss} \equiv prob\{s_{t+1} = s_t \mid s_t = s\},\ s \in \{L, H\}$.

State histories are represented by $s^t = (s_0, ..., s_t)$. For every household, the long-run probability of state $s \in \{L, H\}$ is

$$\pi_H = \frac{1-\pi_{LL}}{2-\pi_{LL}-\pi_{HH}},\ \pi_L = 1-\pi_H \qquad (1)$$

The asymptotic values (π_H, π_L) are steady states for the probabilities (π_τ^H, π_τ^L) of being in state H or L after τ periods, conditional on the current state. From the Markov transition matrix, we obtain

$$\pi_\tau^H = \pi_{HH}\ \pi_{t-1}^H + (1-\pi_{LL})\,(1-\pi_{t-1}^H)$$

which leads to the difference equation

$$\pi_\tau^H = \rho_C\ \pi_{\tau-1}^H + 1 - \pi_{LL} \qquad (2a)$$

Here the correlation coefficient

$$\rho_C \equiv \pi_{HH} + \pi_{LL} - 1 \in (-1, 1) \qquad (2b)$$

describes how persistent the Markov process is.
It is easy to check that equation (1) describes the unique stable steady state of the difference equation (2a).
Here are some additional basic assumptions:

<u>Assumption 3.3</u>: Let $\omega(\alpha, s^t)$ be the endowment of agent $\alpha \in [0,1]$ in history s^t.

Then

$$\omega(\alpha, s^t) = \begin{cases} 1-\alpha \quad (\equiv y_L(\alpha)) \;\; if \;\; s_t = L \\ 1+\dfrac{1-\pi_H}{\pi_H}\alpha \;\; (\equiv y_H(\alpha)) \;\; if \;\; s_t = H \end{cases}$$

This specification of income means that the long-term expected income for each agent α equals 1. Expected income in the long run is

$$\begin{aligned} \overline{\omega}(\alpha) &= \pi_H\left(1+\frac{1-\pi_H}{\pi_H}\alpha\right)+(1-\pi_H)(1-\alpha) \\ &= \pi_H + (1-\pi_H)\alpha + (1-\pi_H)(1-\alpha) = 1 \quad \forall \alpha \end{aligned}$$

<u>Assumption 3.4</u>: The initial distribution of households over idiosyncratic states is the same as the long term distribution.

<u>Assumption 3.5</u>: Households share a common utility function. Household α in history s^t has payoff

$$v(\alpha, s^t) = E\left\{\sum_{j=t}^{\infty}\beta^{j-t}u[c(\alpha, s^j)]\Big| s^t\right\}$$

where $c(\alpha, s^j)$ is the consumption by α in history s^t.

<u>Assumption 3.6</u>: An economy with **idiosyncratic shocks** is populated by a continuum of households indexed $\alpha \in [0,1]$ and distributed on the unit interval according to the p.d.f. G. Each household's history is an independent draw from a common Markov process. Endowments are described in Assumption 1.

<u>Remark</u>. Because this economy has a unit population with average income of 1 unit, aggregate income also equals 1.

<u>Assumption 3.7</u>: An economy with **sectoral shocks** contains two types of individuals, indexed $i = 1, 2$, with unit mass each. Sectoral shocks are indexed $s_t \in \{1, 2\}$ and follow a common, binary, symmetric Markov process such that

$$\Pr\{s_{t+1} = s_t \mid s_t\} = \pi, \quad \forall s_t \in \{1, 2\}$$

Endowments for type $i = 1, 2$ in history s^t are

$$\omega(i, s^t) = \omega(i, s_t) = 1+\alpha \quad if \quad s_t = i$$
$$\qquad\qquad\qquad\quad 1-\alpha \quad if \quad s_t \neq i$$

5. Complete Markets with Perfect Enforcement: A Planning Problem

Planner chooses consumption allocations $c(\alpha, s^t) \in \{c(\alpha, H), c(\alpha, L)\}$ which depend on current events only to maximize a social welfare function subject to an aggregate resource constraint, i.e.,

$$\int_0^{\overline{\alpha}} [\pi_H \; c(\alpha, H) + (1-\pi_H)\; c(\alpha, L)]\, G(d\alpha) \leq 1$$

The social welfare function (SWF) is

$$W = \int_0^{\overline{\alpha}} [\pi_H \; V(\alpha, H) + (1-\pi_H)\, V(\alpha, L)]\, \lambda(\alpha)\, G(d\alpha)\,,$$

where $\lambda(\alpha)$ is the social weight of household α and $V(\alpha, s)$ is the expected value of discounted utility (or continuation payoff) for household α in current state $s = H, L$. Note that a recursive definition of the payoff satisfies

$$V(\alpha, H) = u[c(\alpha, H)] + \beta\, \pi_{HH}\; V(\alpha, H) + \beta\, (1-\pi_{HH})\, V(\alpha, L)\,, \qquad (3a)$$

$$V(\alpha, L) = u[c(\alpha, L)] + \beta\, \pi_{LL}\; V(\alpha, L) + \beta\, (1-\pi_{LL})\, V(\alpha, H)\,. \qquad (3b)$$

To solve equations (3a) and (3b) for $\{V(\alpha, H), V(\alpha, L)\}$, we define ρ_C from equation (2b). (Then $\rho_C = 0$ means that idiosyncratic shocks are i.i.d.; when $\rho_C = 1$, shocks are perfectly positively correlated; when $\rho_C = -1$, shocks are perfectly negatively correlated). Solving equations (3a) and (3b), we obtain

$$(1-\beta)(1-\beta\rho_C)\, V(\alpha, H) = (1-\beta\, \pi_{LL})u[c(\alpha, H)] + \beta(1-\pi_{HH})u[c(\alpha, L)] \qquad (4a)$$

$$(1-\beta)(1-\beta\rho_C)\, V(\alpha, L) = (1-\beta\, \pi_{HH})u[c(\alpha, L)] + \beta(1-\pi_{LL})u[c(\alpha, H)]\,. \qquad (4b)$$

To solve the planner's problem defined above, we look for a saddlepoint of the Lagrangean

$$L = W + \mu\left\{1 - \int_0^{\overline{\alpha}} [\pi_H \; c(\alpha, H) + (1-\pi_H)\; c(\alpha, L)]\lambda(\alpha)G(d\alpha)\; \right\}$$

Set $\partial L / \partial c(\alpha, s) = 0$, for $s \in \{L, H\}$ and obtain

$$\lambda(\alpha)u'[c(\alpha, H)] = \frac{\mu}{1-\beta\; \rho} = \lambda(\alpha)u'[c(\alpha, L)], \quad \forall \alpha \in [0,1].$$

$$\Rightarrow \; c(\alpha, s) = c^*(\alpha), \qquad \forall`s \in \{L, H\}.$$

Therefore, consumption may depend on type and an aggregate income but not on idiosyncratic or sectoral state histories.

Equal-treatment optimum:

If $\lambda(\alpha)=1,\ \forall\alpha$, then $c(\alpha,s)=1,\ \forall(\alpha,s)$, and $V(\alpha,H)=V(\alpha,L)=\dfrac{u(1)}{1-\beta}$.

Question. If households were allowed to deviate permanently from the equal-treatment optimum allocation to autarky, would any household $\alpha\in[0,1]$ ever do so?

Before we answer this question, we duplicate the planner's allocation through competitive markets.

6. Complete Markets with Perfect Enforcement: Competitive Equilibrium

We decentralize the planner's optimal allocation by allowing households to trade Arrow securities contingent on future idiosyncratic or sectoral states $s'=L,H$. Households buy securities contingent on $s'=L$, and sell securities contingent on $s'=H$. The former deliver income when individual endowment is low; the latter promise to give up income when individual endowment is high. Let $p(s,s')$ be the price at $s_t=s$ of security paying 1 unit of consumption at $s_{t+1}=s'$, and paying zero if $s_{t+1}\neq s'$.

Definition 3.1. An equilibrium is a list of security prices $p(s,s')$, security holdings $Q(\alpha,s')$, and consumption allocations $c(\alpha,s)$ for each $\alpha\in[0,1]$ and $(s,s')\in\{L,H\}\times\{L,H\}$ such that

(i) Household α maximizes the continuation payoff at time $t=0$,

$$V(\alpha,s_0)=E\left\{\sum_{t=0}^{\infty}\beta^t u[c(\alpha,s_t)]\ \middle|s_0\right\}$$

under the budget constraints

$$c(\alpha,s)+\sum_{s'}p(s,s')\,Q(\alpha,s')=\omega(\alpha,s)+Q(\alpha,s)$$

given an initial state s_0, prices $p(.)$, and initial security holdings $Q(\alpha,s_0)$.

(ii) Markets clear, that is,

$$\int_0^1\{\pi_H[c(\alpha,H)-\omega(\alpha,H)]+(1-\pi_H)[c(\alpha,L)-\omega(\alpha,L)]\}\,G(d\alpha)=0$$

in the goods market, and

$$\pi_H\int_0^1 Q(\alpha,H)\,G(d\alpha)+(1-\pi_H)\int_0^1 Q(\alpha,L)\,G(d\alpha)=0$$

in the claims market.

The clearing condition in the goods market states that the excess demands of high income people and low income people sum to zero when weighted by their population proportions. The corresponding securities market condition means that the supply of claims contingent on future states of high income is exactly balanced by the demand for claims contingent on future states of low income.

As expected, an equilibrium with perfect enforcement delivers an optimal allocation of resources. One can easily show the following result:

Theorem 3.1. There is a unique equilibrium with complete consumption smoothing. In particular,

(a) $c(\alpha, s) = c^*(\alpha), \ \forall s$

(b) Security prices satisfy $p(s, s') = \beta\pi(s, s')$
[because the f.o.c. is $p(s, s')\, u'(c(\alpha, s)) = \beta\pi(s, s')u'(c(\alpha, s'))$ and equilibrium consumption is state-invariant for all α, that is, $c(\alpha, s) = c(\alpha, s')$.]

(c) Sectoral or idiosyncratic holdings $(Q(\alpha, H), Q(\alpha, L))$ satisfy the budget constraints.

(d) There exists an initial distribution of assets that replicates itself and supports a *symmetric equilibrium* with equal consumption for all agents, i.e., such that $c^*(\alpha) = 1, \ \forall \alpha \in [0,1]$. Specifically,

$$Q^*(\alpha, H) = -\frac{1}{(1-\beta)(1-\beta\,\rho_C)}\left[(1-\beta\pi_{LL})\,\omega(\alpha, H) + \beta(1-\pi_{HH})\,\omega(\alpha, L) + \beta\,\rho_C - 1\right]$$

$$Q^*(\alpha, L) = -\frac{1}{(1-\beta)(1-\beta\,\rho_C)}\left[\beta(1-\pi_{LL})\,\omega(\alpha, H) + (1-\beta\pi_{HH})\,\omega(\alpha, L) + \beta\,\rho_C - 1\right]$$

where $\rho_C \equiv \pi_{HH} + \pi_{LL} - 1 \in [-1, 1]$.

Example 3.1. (Deterministic symmetric economy): Suppose $\pi_{HH} = \pi_{LL} = 0$, i.e., $\rho_C = -1$. Then state histories for the two agents or sectors are $s^t = \begin{cases} H, L, H, L, ... \\ L, H, L, H, ... \end{cases}$, and the asymptotic distribution of states is $(\pi_H, \pi_L) = (1/2, 1/2)$. Assume that endowments are

$$\omega(\alpha, L) \equiv y_L(\alpha) = 1 - \alpha$$

$$\omega(\alpha, H) \equiv y_H(\alpha) = 1 + \alpha\frac{1-\pi_H}{\pi_H} = 1 + \alpha$$

Then Theorem 1(d) says that we can choose an initial distribution of asset holdings

$$Q(\alpha, H) = -\frac{1+\alpha+\beta(1-\alpha)-1-\beta}{1-\beta^2} = -\frac{\alpha}{1+\beta} = -Q(\alpha, L)$$

which generates for each agent a constant consumption stream $c_t^1 = c_t^2 = 1, \forall t$.

This distribution aggregates to zero and replicates itself in equilibrium. The present value of consumption equals $Q(\alpha, s)$ plus the PV of endowment income.
For example, in current state L, the PV is

$$PV(income) + Q = \frac{1-\alpha+\beta(1+\alpha)}{1-\beta^2} + \frac{\alpha}{1+\beta} = \frac{1}{1-\beta}$$

The same is true in current state H. Therefore, $c(\alpha, L) = c(\alpha, H) = 1$. To understand this example, note that the initial holding $Q(\alpha, L)$ is equivalent to adding income $\delta = (1-\beta)\alpha$ to the endowment of low income agents and subtracting the same amount from the endowment of high income agents. In fact, Q_L is the present value of the endowment stream $(\delta, 0, \delta, 0, ...)$. Figure 3 shows how the vector $(c_H, c_L) = (1,1)$ is the common claim of all households $\alpha \in [0,1]$ maximizing utility when $\beta R = 1$ and endowment is $1+\alpha-\delta$ when $s = H$, and $1-\alpha+\delta$ when $s = L$.

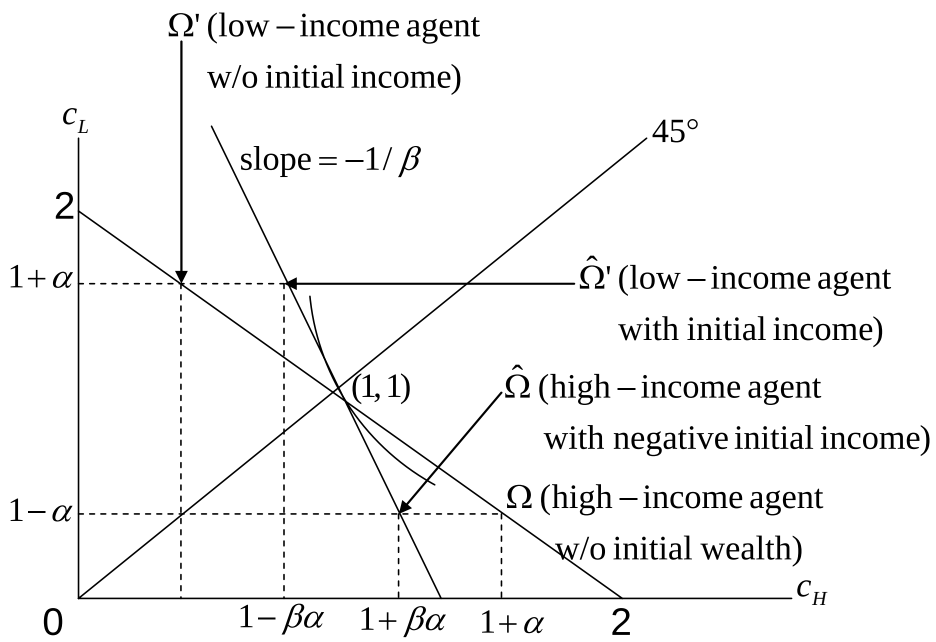


Figure 3

Now we return to the question asked previously.

<u>Question</u>. What if we allow high-state households to deviate from the allocation $c(\alpha, s) = 1$, $\forall(\alpha, s)$, for one period and pay some penalty? Would any household exercise this option?

The answer clearly depends on the economic consequences of deviating from $c(\alpha, s) = 1$. Suppose we specify the following penalty: Let $\phi \in [0,1]$ =exogenous probability of detecting default. Then a detected defaulter goes into perpetual autarky. An undetected defaulter obtains costless debt relief. Under this arrangement, the payoff from solvency is

$$V(\alpha, s) = \frac{u(1)}{1-\beta}, \quad \forall(\alpha, s)$$

Payoff from default for a high-state household is

$$V_D(\alpha, H) = \phi V_A(\alpha, H) + (1-\phi)\frac{u[1+(1-\beta)\alpha/(1+\beta)]}{1-\beta}, \qquad (5a)$$

where, by analogy with equation (3a),

$$(1-\beta\rho_C)(1-\beta)V_A(\alpha, H) = (1-\beta\pi_{LL})u[\omega(\alpha, H)] + \beta(1-\pi_{HH})u[\omega(\alpha, L)] \qquad (5b)$$

Here V_A is the value of autarky for a high-income household and $(1-\beta)\alpha/(1+\beta)$ is the permanent consumption increment for undetected defaulters. The present value of that increment is $\alpha/(1-\beta)$ and equals the value of the forgiven debt $Q_H = -\alpha/(1+\beta)$.

Remember also from equations (4a, 4b) that the payoff from autarky in the high state satisfies equation (5b). In our example, $\rho_C = -1$ and the autarky payoff becomes

$$(1-\beta) V_A(\alpha, H) = \frac{u(1+\alpha) + \beta\, u(1-\alpha)}{1+\beta} \qquad (5c)$$

From (5a) and (5b) it is easy to see that household $\alpha \in [0,1]$ will not deviate from the first-best outcome if, and only if

$$(1+\beta)\, u(1) \geq \phi[u(1+\alpha) + \beta\, u(1-\alpha)] + (1-\phi)(1+\beta)\, u\left(1 + \frac{1-\beta}{1+\beta}\alpha\right). \qquad (6)$$

A particularly interesting special case of (6) is that of perfect monitoring, $\phi = 0$, when (6) reduces to

$$(1+\beta)u(1) \geq u(1+\alpha) + \beta\, u(1-\alpha) \qquad (*)$$

As shown in the following figure, condition (*) is satisfied, and the first-best allocation deminates autarky, iff the endowment point $\Omega = (1+\alpha, 1-\alpha)$ is between E' and F; (*) fails iff Ω is in the interval (EE').

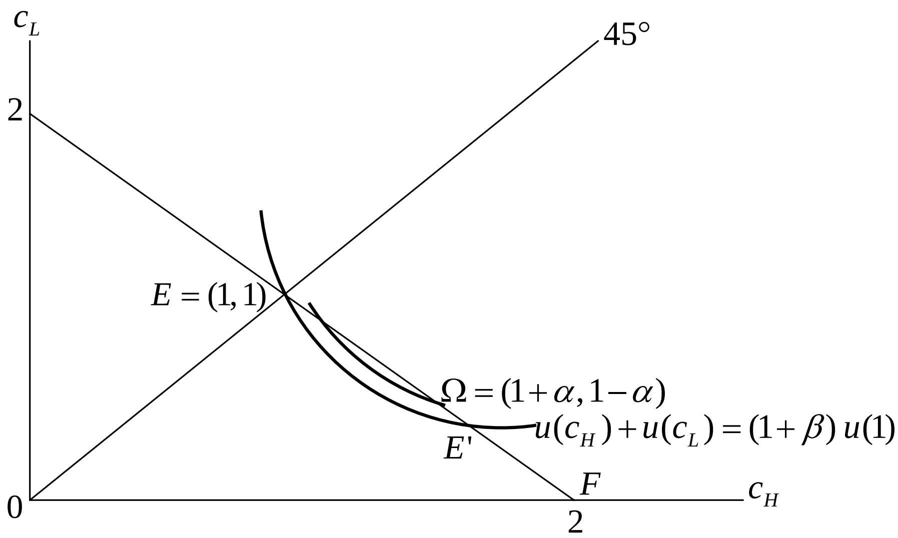


Figure 5

Inspecting Figure 5 reveals that (*) is likely to fail for $\phi = 1$ if $\begin{cases} \beta \text{ is low} \\ \alpha \text{ is low} \\ u(.) \text{ is nearly linear} \end{cases}$

[See also, Kehoe and Levine (1993).]

In addition, if (*) holds for $\phi = 1$, it will still fail for smaller values of ϕ unless ϕ is "large enough," i.e., unless

$$\phi \geq \frac{u(1+\Delta) - \dfrac{u(1+\alpha)+\beta u(1-\alpha)}{1+\beta}}{u(1+\Delta)-u(1)} \qquad (***)$$

where $\Delta = \alpha(1-\beta)/(1+\beta)$, and the concavity of $u(.)$ guarantees that the numerator on the RHS of equation (***) is positive.

<u>Problem 3.1</u>: An economy consists of two groups of agents $i = 1, 2$ with mass 1 each and constant income stream $y_t^i = 1$ for all agents i and time t. Initial assets are zero for all i. Utility functions $u_i = \sum_{t=0}^{\infty} \beta_t^i \log(c_t^i)$ with $0 < \beta_1 < \beta_2 < 1$.

(a) Describe the equilibrium outcome under perfect enforcement. What are the asymptotic values of consumption and of the rate of interest?

(b) Is the perfect enforcement outcome achievable under limited enforcement? If so, for what parameter values?

[Assume default is punished with perpetual autarky with probability one]

Financial Frictions in Dynamic Economics

Class Notes from Econ 589

Costas Azariadis, Fall 2008

Lecture 4: Debt limits in deterministic exchange economies

1. Constrained optimum with limited enforcement and strong default penalties

First we solve the social planner's problem in a deterministic economy with two agents and endowments that fluctuate between two values $1+\alpha$ and $1-\alpha$. The planner's problem is to choose a stationary allocation

$$c_t^i = c_H \in [1, 1+\alpha] \quad \text{if} \quad \omega_t^i = 1+\alpha$$

$$c_t^i = c_L \in [1, 1-\alpha] \quad \text{if} \quad \omega_t^i = 1-\alpha$$

to maximize some well-defined social welfare function (SWF) subject to appropriate resource constraints (RC's) and to participation constraints (PC's).

Agents have the option of deviating from the planner's allocation at any point in time and going into perpetual autarky. The default payoff is consistent with two-sided exclusion from asset markets.

The value of default for agent $i = 1, 2$ is

$$V_D^H = \frac{u(1+\alpha) + \beta\, u(1-\alpha)}{1-\beta^2} \quad \text{if current income is high} \tag{1a}$$

$$V_D^L = \frac{u(1-\alpha) + \beta\, u(1+\alpha)}{1-\beta^2} \quad \text{if current income is low.} \tag{1b}$$

The value of solvency$=(V^H, V^L)$ if the current state is (high, low), where

$$V^H = \frac{u(c_H) + \beta\, u(c_L)}{1-\beta^2} \tag{2a}$$

$$V^L = \frac{u(c_L) + \beta\, u(c_H)}{1-\beta^2} \tag{2b}$$

Assume: *The high-income agent prefers autarky to the Arrow-Debreu allocation*, i.e.,

$$(1+\beta)\,u(1) < u(1+\alpha) + \beta\,u(1-\alpha) \qquad (3)$$

The resource constraint is

$$c_H + c_L \le 2$$

and the participation constraints are $V_D^H \le V^H$ and $V_D^L \le V^L$ or

$$u(c_L) + \beta\,u(c_H) \ge u(1-\alpha) + \beta\,u(1+\alpha) \qquad \text{(PC1)}$$

$$u(c_H) + \beta\,u(c_L) \ge u(1+\alpha) + \beta\,u(1-\alpha) \qquad \text{(PC2)}$$

It is easy to show that (PC2) implies (PC1): Suppose not, that is, (PC2) holds while (PC1) does not. Then

$$u(c_H) + \beta\,u(c_L) - u(c_L) - \beta\,u(c_H) > u(1+\alpha) + \beta\,u(1-\alpha) - u(1-\alpha) + \beta\,u(1+\alpha)$$

$$\Rightarrow u(1-\alpha) - u(c_L) > u(1+\alpha) - u(c_H)$$

which contradicts $c_H \le 1+\alpha$ and $c_L \ge 1-\alpha$.

Finally, the SWF with equal treatment is $V^H + V^L$ or equivalently $u(c_H) + u(c_L)$. Therefore, the planner's problem becomes

$$\max \; u(c_H) + u(c_L)$$

$$s.t. \quad c_H + c_L \le 2 \qquad \text{(RC)}$$

$$u(c_H) + \beta\,u(c_L) \ge u(1+\alpha) + \beta\,u(1-\alpha) \qquad \text{(PC2)}$$

The unique solution is $(c_H, c_L) = (\hat{x}, 2-\hat{x})$ where $\hat{x} \in [1, 1+\alpha]$ is the smallest number satisfying

$$u(x) + \beta\,u(2-x) = u(1+\alpha) + \beta\,u(1-\alpha)$$

with implied interest yield $\hat{R} = \dfrac{u'(\hat{x})}{\beta\,u'(2-\hat{x})} \in \left(1, \dfrac{1}{\beta}\right)$

From Figure 6 it is clear that the allocation $(c_H, c_L) = (\hat{x}, 2-\hat{x})$ is

a) Constrained optimal (it solves the planner's problem)

b) Dynamically efficient (because the PV of aggregate income stream is finite)

<u>Question</u>. Is autarky constrained optimal?

<u>Ans</u>. Yes, if $\dfrac{1}{\beta}\dfrac{u'(1+\alpha)}{u'(1-\alpha)} \equiv \text{autarky yield} \ge 1$, for all agents

No, if $\dfrac{1}{\beta}\dfrac{u'(1+\alpha)}{u'(1-\alpha)} < 1$ (as in Figure 6)

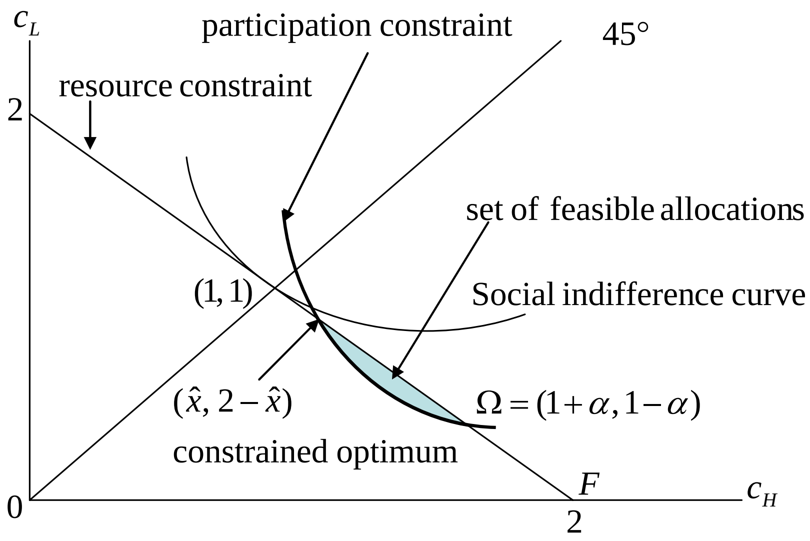


Figure 6: The constrained optimal allocation.

Problem 4.1: Formulate and solve the planner's problem under the assumption that there is a continuum of agents indexed $\alpha \in [0,1]$ with mass 1 and endowment $\omega_t(\alpha) = \alpha$ if $t = 0,2,...$; and $\omega_t(\alpha) = 2-\alpha$ if $t = 1,3,...$. Suppose α is distributed uniformly on $[0,2]$. Assume the planner chooses deterministic allocations $(c_H(\alpha), c_L(\alpha))$ indexed on type and income state.

(a) Show that $c_H(\alpha) > c_L(\alpha)$ for all α, i.e., no agent enjoys completely smooth consumption. [Hint: Assume $c_H(\alpha) = c_L(\alpha)$ for some α and obtain a contradiction]

(b) Prove that agents will be split into two groups. People with highly variable endowments will be rationed (that is, their participation constraint will be binding) when income is high, unrationed when income is low. Agents with low income variability will be completely autarkic.

2. Equilibrium with limited enforcement and strong default penalties

To decentralize efficient allocations as defined in the previous section, we assume:

(1) Two-sided exclusion following default

(2) Symmetric first-best allocation violates the participation constraint for high-income households, that is,

$$(1+\beta)u(1) < u(1+\alpha) + \beta\, u(1-\alpha)$$

Recall also from the previous section that the constrained optimal planner's allocation is

$$c_t^i = \begin{cases} \hat{x} & \text{if } \omega_t^i = 1+\alpha \\ 2-\hat{x} & \text{if } \omega_t^i = 1-\alpha \end{cases}$$

where $\hat{x} \in [1, 1+\alpha]$ is the smallest solution to

$$u(x) + \beta\, u(2-x) = u(1+\alpha) + \beta\, u(1-\alpha)$$

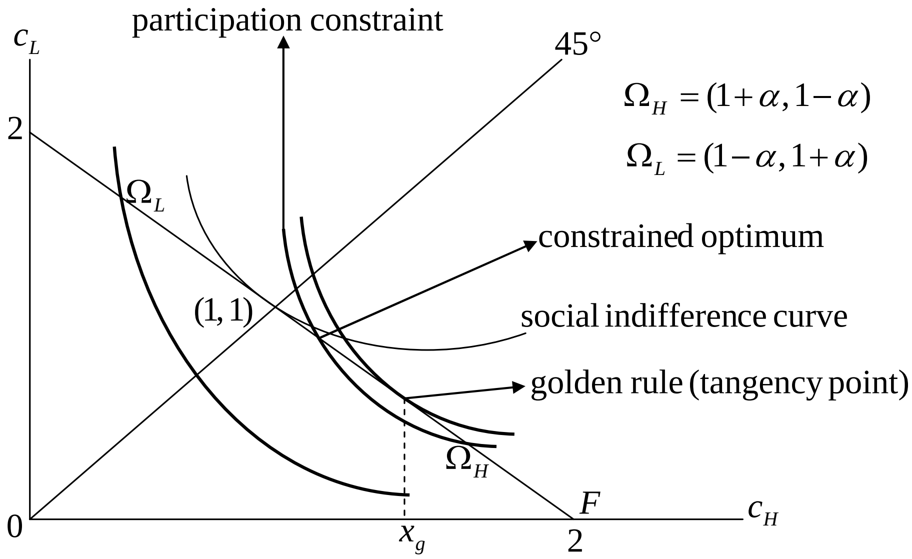


Figure 7

Planner's problem can thus be written as

$$\begin{aligned} &\max \; u(c_H) + u(c_L) \\ &\text{s.t.} \quad c_H + c_L \le 2 \\ &\qquad\; u(c_H) + u(c_L) \ge u_A \equiv u(1+\alpha) + \beta\, u(1-\alpha) \end{aligned} \qquad \text{(P)}$$

or

$$\begin{aligned} &\max_{x \in [0,2]} \; u(x) + u(2-x) \\ &\text{s.t.} \quad u(x) + u(2-x) \ge u_A \end{aligned} \qquad \text{(P')}$$

Note: $\hat{x} = 1+\alpha$ iff $1+\alpha < x_g$ where the implied interest yield at x_g equals the zero growth rate of aggregate income or $u'(x_g) = \beta\, u'(2-x_g)$. This solution is called the *golden rule* allocation.

Suppose next that the only asset is loans in zero supply. Then we define competitive equilibrium as follows.

Definition 4.1. The sequence $(c_t^1, c_t^2, b_t^1, b_t^2, L_t^1, L_t^2, R_t)$ is a competitive equilibrium of the economy with two-sided exclusion if

a) Consumer i solves the following problem:

$$\max \sum_{t=0}^{\infty} \beta^t \, u(c_t^i)$$
$$s.t. \quad c_t^i + b_t^i = \omega_t^i + R_{t-1} \, b_{t-1}^i$$
$$-b_t^i \le L_t^i, \quad \forall (t, i)$$

taking (R_t, L_t^i) as exogenous.

b) Markets clear, that is,

(goods) $\quad c_t^1 + c_t^2 = \omega_t^1 + \omega_t^2 = 2$

(assets) $\quad b_t^1 + b_t^2 = 0$

c) Debt limits (L_t^1, L_t^2) are the largest values consistent with solvency payoffs never falling below default payoffs, that is,

$$V_t^S - V_t^D \equiv \sum_{s=0}^{\infty} \beta^s [u(c_{t+s}^i) - u(\omega_{t+s}^i)] \ge 0, \quad \forall (i, t).$$

Next we show that the constrained optimum and autarky are both stationary equilibria. In particular, it is easy to show

Lemma 4.1. The constrained optimum allocation can be decentralized as a stationary competitive equilibrium with

$$c_t^1 = \begin{cases} \hat{x} & if \quad \omega_t^i = 1 + \alpha \\ 2 - \hat{x} & if \quad \omega_t^i = 1 - \alpha \end{cases}$$

$R_t = \hat{R}$, where $u'(\hat{x}) = \beta \, \hat{R} \, u'(2 - \hat{x})$, and $\hat{R} \in \left(1, \dfrac{1}{\beta}\right)$.

$$b_t^i = \begin{cases} \hat{L} & if \quad \omega_t^i = 1 + \alpha \\ -\hat{L} & if \quad \omega_t^i = 1 - \alpha \end{cases}$$

where $\hat{L}$ satisfies the budget constraint

$$\hat{x} + \hat{L} = 1 + \alpha - \hat{R}\,\hat{L} \quad \Rightarrow \quad \hat{L} = \frac{1+\alpha-\hat{x}}{1+\hat{R}}.$$

The next result shows how to decentralize autarky.

<u>Lemma 4.2</u>. Autarky is always an equilibrium with

$$c_t^i = \omega_t^i, \quad \forall (i,t)$$

$$L_t^i = 0 \quad \text{if} \quad \omega_t^i = 1-\alpha$$

$$R_t = \frac{u'(1+\alpha)}{\beta\, u(1-\alpha)} \equiv \overline{R}, \quad \forall t$$

In general, dynamical equilibria are sequences (x_t) such that $(c_t^H, c_t^L) = (x_t, 2-x_t)$ and x_t satisfies

(a) $1 \le x_t \le 1+\alpha$ (incomplete consumption smoothing)

(b) $u'(x_t) = \beta\, R_t\, u'(2-x_{t+1})$ (the consumption-Euler equation for the high-income household)

(c) $u(x_t) + \beta\, u(2-x_{t+1}) = u_A$ (the participation constraint)

(d) If $\overline{R} \equiv \dfrac{u'(1+\alpha)}{\beta\, u'(1-\alpha)} > 1$, then autarky $x_t = 1+\alpha, \forall t$ is the only equilibrium and it is Pareto optimal.

In this case, both welfare theorems hold. Equilibrium is supported by the price $R_t = \overline{R}, \forall t$ and debt limits $L_t = 0$

(e) If $\overline{R} < 1$, then we have two steady states $x_t \in \{\hat{x}, 1+\alpha\}$, $\forall t$, plus a continuum of sequences (x_t) that start at $x_0 \in (1, 1+\alpha)$ and converge to autarky, $x_\infty = 1+\alpha$. Allocations are Pareto-ranked depending on the initial value x_0. In particular, the allocation $x_t = \hat{x}, \forall t$ is constrained optimal. All other equilibria are suboptimal. *The first welfare theorem fails in this case but the second welfare theorem holds*.

(f) Collateral borrowing vs reputational borrowing: "autarky" should be understood as zero reputational borrowing. If we had allowed collateral, all actual borrowing would have been against collateral.

(g) Why is reputational borrowing indeterminate? A *dynamic complementarity* exists between current and expected future debt limits. Higher future debt limits raise the value of market participation today, and also the penalty for default which helps enlarge current debt limits. Dynamic equilibria are graphed in Figure 8.

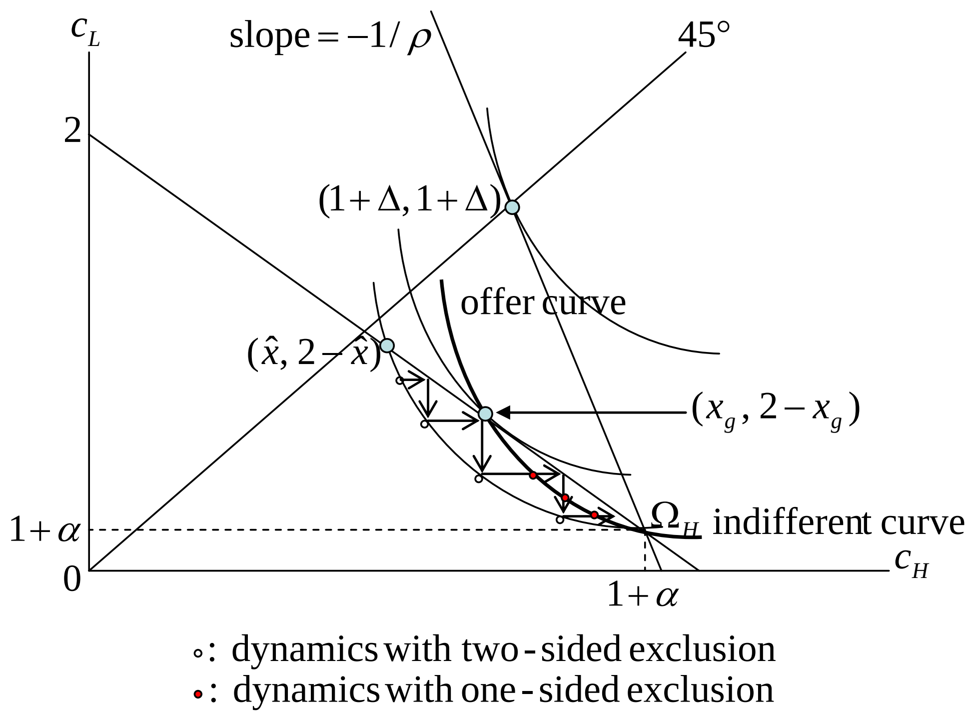


Figure 8. Dynamic equilibria with two-sided and one-sided exclusion.

3. Equilibrium with a weak default penalty

Bulow-Rogoff (1989) and Hellwig-Lorenzoni (2007) explore equilibria in a limited enforcement economy which punishes default by banning offenders from ever borrowing again but permitting them to lend freely at the market interest rate. In this environment the Arrow-Debreu allocation $c_H = c_L = 1$ clearly violates the participation constraint. To see this, check that inequality (6) in lecture 3 fails for $\phi = 1$ and all parameter values (α, β). Similarly, the allocation achievable by a defaulting high-income agent at the Arrow/Debreu interest rate $1/\beta$, $c_H = c_L = 1+\alpha(1-\beta)/(1+\beta)$, dominates $c_H = c_L = 1$, as shown in Figure 8.

Competitive equilibrium in this setting is defined analogously with the previous section: it is any sequence (x_t) such that $(c_t^H, c_t^L) = (x_t, 2-x_t)$, with x_t satisfying

(a) $1 \le x_t \le 1+\alpha$ (potentially incomplete consumption smoothly)

(b) $u'(x_t) = \beta\, R_t\, u'(2-x_{t+1})$ (consumption-Euler equation)

(c) $u(x_t) + \beta\, u(2-x_{t+1}) = \max_{s\in[0,1+\alpha]} \{u(1+\alpha-s) + \beta u(1-\alpha+R_t s)\}$

(indifference between default and solvency)

Clearly,

$$x_t = 1+\alpha \quad \text{if} \quad R_t = \overline{R} = u'(1+\alpha)/[\beta u'(1-\alpha)] < 1$$
$$x_t = x_g \quad \text{if} \quad R_t = 1$$

Thus, x_t lies on the *offer curve through the initial endowment point*, as shown in Figure 8. Dynamical equilibria are exactly like those of a pure exchange overlapping generations economy with constant population, constant money supply, a two-period lifecycle and a lifecycle endowment $\omega = (1+\alpha, 1-\alpha)$.

For example, there exists a *golden rule* steady state $(c_H, c_L) = (x_g, 2-x_g)$ which permits less consumption smoothing than the constrained-efficient allocation $(c_H, c_L) = (\hat{x}, 2-\hat{x})$ does under two-sided exclusion. In addition, we have a constrained inefficient autarky state $(c_H, c_L) = (1+\alpha, 1-\alpha)$, and a continuum of equilibria that start with $x_0 \in (x_g, 1+\alpha)$ and converge to autarky.

The only difference between infinitely-lived agents with one-sided exclusion and overlapping generations is in the type of assets that are needed to support the same competitive allocation. An asset in positive net supply like public debt or fiat money is useful in the lifecycle economy to overcome the impossibility of loans between young and old households. Private debt achieves the same goal in an infinite lifecycle economy with one-sided exclusion without any help from public institutions.

Financial Frictions in Dynamic Economics

Class Notes from Econ 589

Costas Azariadis, Fall 2008

Lecture 5: Competitive equilibrium in stochastic exchange economies with limited enforcement

This lecture extends the material in lecture 4 to stochastic endowment economies. Alvarez and Jermann (2000) is a good reference for this material.

1. Introduction

Assume that default is punished by perpetual autarky. Let $c(i, s^t)$ =consumption for agent i in history s^t

$b(i, s^{t+1})$ =security holdings for i in history s^t contingent on future state s_{t+1}

$p(s^t, s_{t+1})$ =Arrow security price at history s^t for 1 unit of consumption in future state s_{t+1}

$L(i, s^{t+1})$ =debt limit such that $-b(i, s^{t+1}) \leq L(i, s^{t+1})$ for all (i, s^{t+1})

Definition 5.1. A competitive equilibrium is a list $\{c(i, s^t), b(i, s^t), p(i, s^t), L(i, s^t)\}$ such that

a) Given prices and debt limits, agent $i = 1, 2$ maximizes the payoff conditional on a given initial state s_0 and subject to budget constraints and debt limits. The formal problem is

$$\max \; V(i, s_0) = E\left\{\sum_{t=0}^{\infty} \beta^t u[c(i, s^t)] \middle| s_0 \right\}$$

$$s.t. \quad -b(i, s^{t+1}) \le L(i, s^{t+1}),$$

$$c(i, s^t) + \sum_{s_{t+1}} p(s^t, s_{t+1})\, b(i, s^t, s_{t+1}) \le \omega(i, s_t) + b(i, s^t)$$

b) Markets clear, that is,

(goods) $\quad c(1, s^t) + c(2, s^t) = 2\,, \quad \forall s^t$

(assets) $\quad b(1, s^{t+1}) + b(2, s^{t+1}) = 0\,, \quad \forall s^{t+1} \in \{(s^t,1), (s^t,2)\}$

c) Debt limits are the largest values satisfying $V(i, s^t) \ge V_A(i, s^t)$, that is, ones that prevent the solvency payoff from falling below the default payoff for all agents and all possible histories.

2. Autarky

Idea: financial autarky is always an equilibrium supported by

(i) prices matching the MRS of the most patient agent (highest β, lowest rate of income growth); and

(ii) zero debt limits for all other agents

Theorem 5.1. Autarky is always an equilibrium supported by security prices

$$p(s, s') = \max_i \left\{ \beta\, \pi(s, s') \frac{u'(\omega(i, s'))}{u'(\omega(i, s))} \right\}$$

and debt limits

$$L(i, s^{t+1}) = \begin{cases} 0 & \text{if} \quad s_{t+1} = i \\ \infty & \text{if} \quad s_{t+1} \ne i \end{cases}$$

Autarky security prices $p(s, s')$ are generally higher than the prices $p^*(s, s')$ supporting he Arrow/Debreu equilibrium because security suppliers are rationed. In particular,

$$p(1,1) = p(2,2) = \beta\,\pi = p^*(1,1) = p^*(2,2)$$

$$p(1,2) = p(2,1) = \beta\,(1-\pi)\frac{u'(1-\alpha)}{u'(1+\alpha)} > \beta\,(1-\pi) = p^*(1,2) = p^*(2,1)$$

Therefore, $p(s,s') \ge p^*(s,s')$, $\forall (s,s')$.

Remark 5.1. Interpret "autarky" as borrowing against collateral only.

Remark 5.2. The autarky risk-free yield is the reciprocal of the total cost of a safe portfolio paying off one consumption unit in every future state. Thus

$$R_F^a \equiv \frac{1}{p(1,1)+p(2,2)} = \frac{1}{\beta\left[\pi + (1-\pi)\dfrac{u'(1-\alpha)}{u'(1+\alpha)}\right]} < \frac{1}{\beta}$$

For $\alpha = 0.1$, $u(\mathrm{c}) = \log(\mathrm{c})$, $\beta = 0.96$,

$$R_F^a = \begin{cases} 1.02 & \pi = 0.9 \\ 0.93 \quad if & \pi = 0.5 \\ 0.85 & \pi = 0 \end{cases}$$

If α is a continuous parameter on $[0,1]$, then $p(1,2) = p(2,1) \to \infty$, and $R_F^a \to 0$ as $\alpha \to 1$. The rate of return on a riskless portfolio falls to -100%!

3. Non-autarkic equilibria

In the deterministic economy with two-sided exclusion, these equilibria were solutions to the participation constraint

$$u(x_t) + \beta\,u(2 - x_{t+1}) = u(1+\alpha) + \beta\,u(1-\alpha) \quad \text{for} \quad 1 \le x_t \le 1+\alpha\,.$$

We generalize this result to a symmetric Markov economy with two types of agents and probability $\pi \in [0,1]$ of remaining in the same state.

Theorem 5.2. Let $(c_t^H, c_t^L) = (x_t, 2 - x_t)$ describe the consumption of the two agents at t. Then competitive equilibria for an economy with two-sided exclusion are solutions to the equation

$$u(x_t) + \beta\,(1-\pi)\,u(2-x_{t+1}) - \beta\,\pi\,u(x_{t+1}) \\ = (1-\beta\pi)\,u(1+\alpha) + \beta\,(1-\pi)\,u(1-\alpha) \qquad (*)$$

with the following properties

(a) $x_t = 1+\alpha, \forall t$ is an equilibrium

(b) If autarky is a constrained optimal allocation, then $x_t = 1+\alpha$, $\forall t$ is the only solution to (*).

(c) If autarky is not a CO allocation, then there is another steady state of (*) such that $x_t = \hat{x} \in (1, 1+\alpha)$ which is CO and unstable.

(d) If autarky is not a CO allocation, then there is a continuum of equilibria indexed on $x_0 \in (1, 1+\alpha)$ all of which converges to autarky. All such equilibria can be Pareto-ranked by the initial value x_0.

(e) Security prices reflect the valuation of unrationed agents, i.e., of those households with the strongest desire for future consumption. Specifically,

$$p_t(1,1) = p_t(2,2) = \beta\,\pi\,\frac{u'(2-\hat{x}_{t+1})}{u'(2-\hat{x}_t)} > \beta\,\pi\,\frac{u'(\hat{x}_{t+1})}{u'(\hat{x}_t)}$$

$$p_t(1,2) = p_t(2,1) = \beta\,(1-\pi)\frac{u'(2-\hat{x}_{t+1})}{u'(\hat{x}_t)} > \beta\,(1-\pi)\frac{u'(\hat{x}_{t+1})}{u'(2-\hat{x}_t)}$$

To prove this theorem, readers are urged to solve the following problems.

Problem 5.1. (a) Define the planner's problem for a stochastic exchange economy with a symmetric two-state Markov matrix and two types of agents.

(b) Suppose the Arrow-Debreu outcome violates the participation constraint. Prove part (b) of Theorem 5.2. Under what conditions is autarky constrained optimal?

(c) Suppose autarky is not constrained optimal. Prove part (c) of Theorem 5.2.

Problem 5.2. Prove parts (d) and (e) of Theorem 5.2 and show that the security prices described in Theorem 5.2 (e) converge to the autarkic prices of Theorem 5.1 for all initial values $x_0 \in (\hat{x}, 1+\alpha)$.

Financial Frictions in Dynamic Economics

Class Notes from Econ 589

Costas Azariadis, Fall 2008

Lecture 6: Growth with limited capital mobility: Financial autarky and irreversible investment

1. Introduction: Capital mobility and economic growth

To understand the importance of capital mobility for long-run economic performance, think of economies which must reallocate capital away from sectors that are temporarily unproductive. A "sector" here may mean an individual firm, an industry or an entire nation inside an integrated world economy. Economies with high capital mobility are able to exploit reversible and temporary differences in TFP between sectors to raise their wealth and consumption.

The term "capital mobility" is purposely left vague because it may mean different things in different contexts. Below we study two distinct examples of capital immobility. The first is financial autarky which prevents all capital movement among sectors, by loans or any other means. Autarky gives us an upper bound to the economic value of financial markets. [cf. Azariadis and Kaas (2007)]

Irreversible investment is our second example which describes an extreme form of adjustment costs. The idea here is that capital installed in one sector cannot be used as consumption or to produce in another sector; see, for example, Sargent (1980) and Olsen (2005). To fix notation, we study an economy with two sectors or two groups of agents. Individuals are both producers and consumers experiencing reversible fluctuations in the productivity of their individual AK technologies. They are indexed $i = 1, 2$ with mass=1 for each type. There is also a Markovian aggregate state $s = 1, 2$ with transition probability $\pi = pr\{s_{t+1} = s \mid s_t = s\}, s = 1, 2$. The transition matrix is $\begin{pmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{pmatrix}$. Consumption is $c(i, s^t)$ and the continuation payoff conditional on history s^t is

$$v(i, s^t) = E\{\sum_{j=0}^{\infty} \beta^j \log[c(i, s^{t+j})] \mid s^t\}$$

The production technology is $y(i, s^t) = A(i, s_t)\, k(i, s^t)$, where $s^t = (s^{t-1}, s_t)$ and

$$A(i, s) = \begin{cases} A & \text{if} \quad i = s \\ \alpha A & \text{if} \quad i \neq s \end{cases} \qquad 0 < \alpha < 1$$

Lilien (1982) called these temporary changes in relative factor productivities "the sectoral shift hypothesis" and attempted to document empirically that sectoral shifts raised aggregate unemployment.

Suppose $y(i, s^t)$ includes undepreciated capital. Note from the agent's consumption-Euler equation that, for a logarithmic utility function, we have the following simple division of resources into consumption and saving:

$$k(i, s^{t+1}) = \beta\, y(i, s^t)$$

$$c(i, s^{t+1}) = (1-\beta)\, y(i, s^t)$$

2. Financial autarky

In this economy no capital can move between sectors. The aggregate growth rate fluctuates between high values, which coincide with capital being mostly employed in the productive sector, and low values which coincide with the opposite state of affairs. Then current output y and future capital k' both depend on the current distribution variable x, that is, on the share of aggregate capital employed in the productive sector. We define future aggregate output= $y' > 0$; current aggregate capital= $k > 0$, and future aggregate capital= $k' > 0$. Shares of total capital used by productive agents now and in the future are $(x, x') \in (0,1)^2$. Then we have

$$y = \frac{1}{2}\left[A\,x\,k + \alpha\,A\,(1-x)\,k\right]$$

$$k' = \beta\,y$$

$$x' = \frac{\text{efficient firm's capital tomorrow}}{\text{all capital tomorrow}}$$

Then with probability π, we have

$$x' = \frac{\beta Axk/2}{\beta y} = \frac{x}{x+\alpha(1-x)},$$

and with probability $1-\pi$, we have

$$x' = \frac{\beta\alpha A(1-\alpha)k/2}{\beta y} = \frac{\alpha(1-x)}{x+\alpha(1-x)}$$

To sum up, the law of motion for the share $x \in [0,1]$ satisfies

$$x' = \begin{cases} f(x;\alpha) \equiv \dfrac{x}{x+\alpha(1-x)} & \text{with probability } \pi \\ h(x;\alpha) \equiv 1 - f(x;\alpha) & \text{with probability } 1-\pi \end{cases} \qquad (1)$$

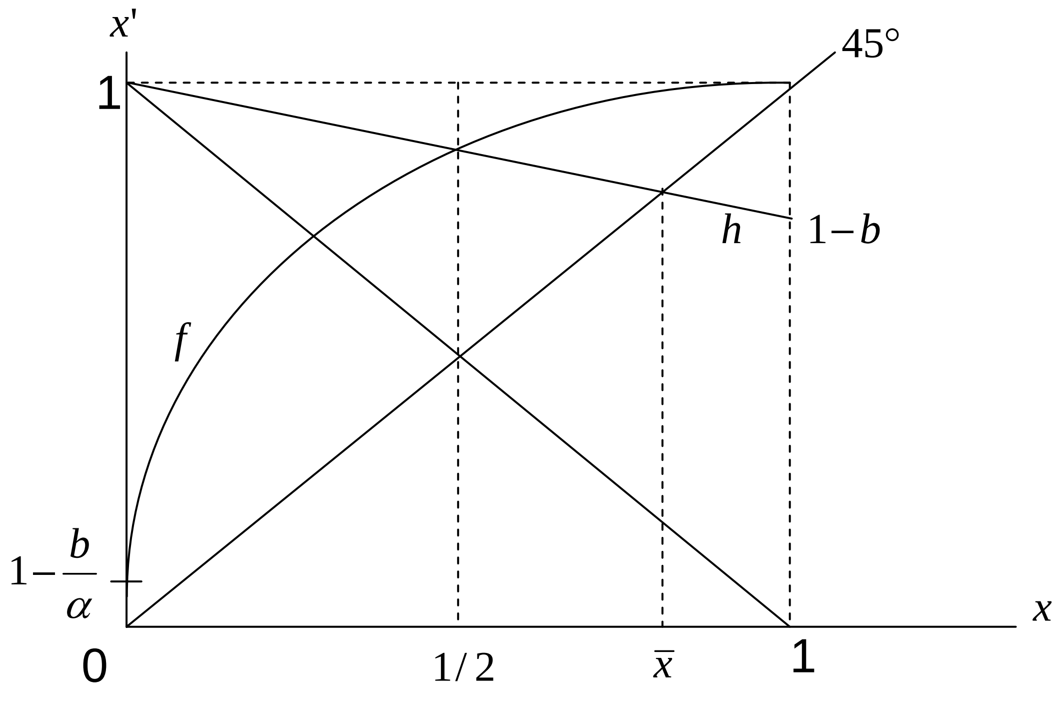


Figure 9

We solve equation (1) for the special case: $\pi = 0$ (deterministic economy)

Then $x_{t+1} = h(x_t; \alpha)$, where $h^2 \equiv h(h(x_t; \alpha)) = \dfrac{\alpha(1-h)}{h + \alpha(1-h)} = x$

The general solution is a cycle with period 2, that is,

$$x_t = \begin{cases} x_0 & t = 0, 2, \ldots \\ h(x_0; \alpha) & t = 1, 3, \ldots \end{cases} \qquad \text{if} \quad x_0 \neq \frac{\sqrt{\alpha}}{1+\sqrt{\alpha}};$$

However, in the special case $x_0 = \sqrt{\alpha}/(1+\sqrt{\alpha})$, we have a steady state $x = x_0, \forall t$.

Under a perfect financial market, this economy would have a balanced growth path $y_t = y_0(\beta A)^t$. With no financial market, the normalized growth rate of aggregate output instead is

$$n = \frac{y'/y}{\beta A} = \frac{Ak'[x' + \alpha(1-x')]}{\beta A y}$$
$$= \alpha + (1-\alpha)x' < 1$$
$$= \begin{cases} \alpha + (1-\alpha)g(x) & \text{with probability } \pi \\ \alpha + (1-\alpha)h(x) & \text{with probability } 1-\pi \end{cases}$$

Aggregate factor productivity is captured by the Solow residual

$$\frac{y}{k} = A[x + \alpha(1-x)]$$

where $[x + \alpha(1-x)] < 1$ is a correction for financial autarky.

<u>Problem 6.1</u>: in the worst case scenario outlined previously, compare aggregate economic performance with sectoral economic performance. In particular, describe how:

(a) The normalized aggregate growth rate n is correlated with the corresponding sectoral growth rates.

(b) Deviations of aggregate output from trend (i.e., its own conditional expectation) are related to deviations of sectoral outputs from trend values.

3. Irreversible investment

Suppose we entrust the economy described in section 1 in the hands of a powerful planner who must assign consumption and capital to the inhabitants of each sector under the constraint that physical capital, once installed, cannot be shifted across sectors or converted into consumption. Except for this constraint, the planner may impose on households any technically feasible allocation she chooses. The planner selects consumption allocations (c_t^H, c_t^L) for the high-productivity and low-productivity agents, and in addition, the corresponding vector of capital stocks (k_t^H, k_t^L) to maximize a typical equal-treatment social welfare function

$$SWF = \sum_{t=0}^{\infty} \beta^t [u(c_t^H) + u(c_t^L)]$$

subject to given initial stocks (k_0^H, k_0^L), the resource constraint

$$c_t^H + c_t^L + k_{t+1}^H + k_{t+1}^L = A\, k_t^H + \alpha\, A\, k_t^L \qquad (2a)$$

and the irreversibility constraints

$$k_{t+1}^H \geq (1-\delta) k_t^L, \quad k_{t+1}^L \geq (1-\delta) k_t^H \qquad (2b)$$

if the state remains the same $(s_{t+1} = s_t)$. If the state changes, the constraints in (2b) are replaced by

$$k_{t+1}^H \geq (1-\delta) k_t^H, \quad k_{t+1}^L \geq (1-\delta) k_t^L \qquad (2c)$$

If investment *were* reversible, the planner would shift all capital to the most productive technology. This means that the first constraint in (2b) and (2c) will never bind while the second always will. From equations (2b), (2c), we obtain

$$(1-x')k' = \begin{cases} (1-\delta)xk & \text{with probability } 1-\pi \\ (1-\delta)(1-x)k & \text{with probability } \pi \end{cases}$$

where, as in section 2

$$k' = \beta\, y \qquad \text{from the consumption - Euler equations}$$

$$= \beta Ak[x + \alpha(1-x)] \qquad \text{from adding up sector outputs}$$

Substituting this equation into the previous one, we obtain

$$x' = \begin{cases} f(x;\alpha,b) \equiv 1 - \dfrac{b\,(1-x)}{x+\alpha(1-x)} & \text{with probability } \pi \\ h(x;\alpha,b) \equiv 1 - \dfrac{b\,x}{x+\alpha(1-x)} & \text{with probability } 1-\pi \end{cases}$$

where

$$b \equiv \frac{1-\delta}{\beta A} \qquad (3b)$$

To ensure that the functions (f, h) map the interval $[0,1]$ into itself, we assume $b \le \alpha$. Note also that

i. $f(0;\alpha,b) = 1 - \dfrac{b}{\alpha}$, $f(1;\alpha,b) = 1$

ii. $h(0;\alpha,b) = 1$, $h(1;\alpha,b) = 1-b$

iii. f is increasing, concave and g is decreasing, convex. In particular, derivatives with respect to x are

$$f_x(x;\alpha,b) = \frac{b}{[x+\alpha(1-x)]^2}, \quad g_x(x;\alpha,b) = -\frac{\alpha\, b}{[x+\alpha(1-x)]^2}$$

iv. Furthermore,

$$g_x(x;\alpha,b) \in (-1,0) \quad \forall x \in [0,1]$$

$$f_x(x;\alpha,b) \in (0,1) \quad \text{if} \quad b < \alpha^2$$

Figure 10 graphs the dynamical system of equations (3a) which looks similar to equation (1) and Figure 9. In particular, we can show (in Problem 6.3) that the solution sequence x_t to equation (3a) will converge to the invariant interval $I = [1-b, 1]$ and will cycle inside I, except in the deterministic case $\pi = 0$ where the economy (see Problem 6.2) converges to the steady state $\bar{x} \in (1/2, 1)$ as shown in Figure 10.

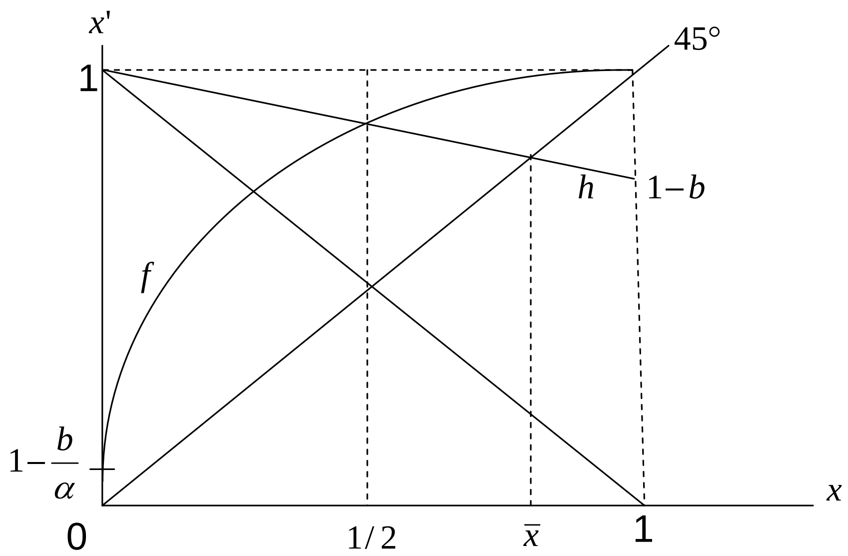


Figure 10. Irreversible investment cycles

Problem 6.2. Describe equilibria for a deterministic economy with irreversible investment. Specifically, show that equilibria converge via damped oscillations to the fixed point of the map $h(x;\alpha,b)$ defined in equation (3a).

Problem 6.3. In the stochastic economy with irreversible investment, show that equilibria will converge to the invariant interval $I=[1-b,1]$. Specifically, show that

(a) $x_t \in I$ implies that with probability 1 that $x_s \in I$ for all $s>t$.

(b) If x_t is outside I, then $x_s \in I$ for some finite $s>t$ with probability 1.

Financial Frictions in Dynamic Economics

Class Notes from Econ 589

Costas Azariadis, Fall 2008

Lecture 7: Collateral borrowing in economic growth

1. Introduction

We saw in Lecture 6 how economies with non-existent or poor financial markets tend to grow slowly and fluctuate a lot about trend even if their PPF is stable. We adapt the Kiyotaki (1998) model of secured borrowing to explore whether the same is true of economies with relatively good financial markets in which all loans are backed by collateral. The general setting is the sectoral shift economy described in Lecture 6 with two sectors and relative productivity shocks. Producers are allowed to borrow up to a fraction $\lambda \in [0,1]$ of the present value of next period's gross output, including undepreciated capital. As before, we use as our state variable the fraction $x \in [0,1]$ of total wealth or equity owned by the productive agent (the "borrower") and by $1-x$ the wealth fraction belonging to the less productive agent (the "lender"). We also denote

$\theta(x) =$ borrower's maximum debt-to-equity ratio

$R(x) =$ yield on loans

$\tilde{R}(x) =$ yield on equity

Clearly, the credit market cannot clear unless

$$\alpha A \le R(x) \le A \tag{1}$$

that is, unless the borrower desires to borrow and the lender wants to lend. Also, for any value $R(x) < A$, all borrowers will express infinite demand for loans and will be rationed. The equity yield exceeds the loan yield by the amount of leverage, i.e.,

$$\tilde{R}(x) = R(x) + \theta(x)[A - R(x)] \tag{2}$$

The debt limit is defined by the equality between what is owed and the value of future collateral. For one unit of borrower wealth, we have

$$R(x)\,\theta(x) = \lambda\,A(1 + \theta(x))$$

which means

$$\theta(x) = \frac{\lambda\,A}{R(x) - \lambda\,A} \tag{3}$$

2. Equilibrium

We describe equilibrium by the law of motion for x, the borrower's share in total wealth, and by the loan yield $R(x)$. First, note that the loan market clears if

$$\theta(x)\,x \le 1 - x \quad \text{and} \quad R(x) = \alpha A \tag{4a}$$

$$\theta(x)\,x = 1 - x \quad \text{and} \quad R(x) \in (\alpha A,\, A) \tag{4b}$$

$$\theta(x)\,x \ge 1 - x \quad \text{and} \quad R(x) = A \tag{4c}$$

Equation (4a) says that the borrower's demand per unit of aggregate wealth must be at most equal to the entire wealth of the lender when the lender's supply of loans is indeterminate. Equation (4c) is a similar statement, with the borrower being indifferent as to loan demand, and the lender lending out his entire wealth.

The law of motion for x is easily derived once we understand that everyone saves a fixed fraction β of their equity. For example, if the state of nature does not change from this period to the next, then

$$x' = f(x;\alpha) \equiv \frac{\beta\,\tilde{R}(x)\,x}{\beta[\,\tilde{R}(x)\,x + R(x)(1 - x)]}$$

If the state changes, then we have

$$x' = h(x;\alpha) = 1 - f(x;\alpha)$$

Combining these two, we obtain

$$x' = \begin{cases} f(x;\alpha) = \dfrac{\tilde{R}(x)\, x}{\tilde{R}(x)\, x + R(x)(1-x)} & \text{with probability } \pi \\ h(x;\alpha) = 1 - f(x;\alpha) & \text{with probability } 1-\pi \end{cases} \quad (5)$$

Next, we substitute equation (3) into (2), and (4a)-(4c) to obtain expression for $\tilde{R}(x)$ and $R(x)$, i.e.,

$$(R(x), \tilde{R}(x)) = \begin{cases} \left(1, \dfrac{1-\lambda}{\alpha-\lambda}\right)\alpha A & \text{if} \quad x \le 1\text{-}\dfrac{\lambda}{\alpha} \\ \left(\dfrac{\lambda}{1-x}, \dfrac{1-\lambda}{x}\right) A & \text{if} \quad x \in \left[1\text{-}\dfrac{\lambda}{\alpha}, 1-\lambda\right] \\ (1,1)\, A & \text{if} \quad x \in [1-\lambda, 1] \end{cases} \quad (6)$$

Figure 11 illustrates

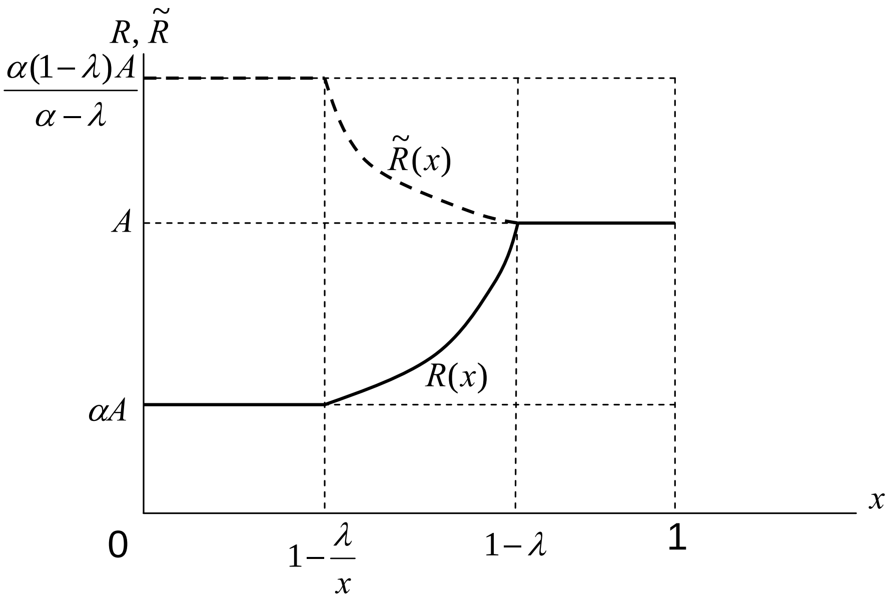


Figure 11. Debt and equity returns

Equation (6) connects the cost of loans to (x, λ). In general, high values of (x, λ) mean that borrowers are able to borrow large amounts of funds. Specifically, for low values of borrower wealth, both the demand for loans and the loan yield are so weak that lenders are indifferent between lending and producing. The economy misallocates its capital and produces inefficiently with a low average value for factor productivity, and a low overall growth rate. Things improve at higher values of x as higher loan yields are a signal for all capital to move to the productive sector. However, the allocation of consumption remains inefficient as long as borrowers and lenders face different yields. Only at $x \geq 1-\lambda$ do the two yields become identical and so do the consumption patterns of lenders and borrowers.

The low of motion for the borrower's equity share is embodied in equations (5) and (6). Substitute the latter into the former to get

$$f(x;\alpha) = \begin{cases} \dfrac{(1-\lambda)x}{\alpha-\lambda+(1-\alpha)x} & \text{if} \quad x \geq 1-\dfrac{\lambda}{x} \\ 1-\lambda & \text{if} \quad x \in \left[1-\dfrac{\lambda}{x}, 1-\lambda\right] \\ x & \text{if} \quad x \in [1-\lambda, 1] \end{cases}$$

Then dynamic equilibria satisfy equation (5):

$$x' = \begin{cases} f(x;\alpha) & \text{with probability } \pi \\ 1-f(x;\alpha) & \text{with probability } 1-\pi \end{cases} \tag{8}$$

3. Deterministic dynamics

Suppose the state s changes for scene each period, and lenders switch roles with borrowers. Then the low of motion for x is given by the map $h(x;\alpha)$ or

$$x_{t+1} = h(x_t;\alpha) \tag{9a}$$

where

$$h(x;\alpha) = \begin{cases} (\alpha-\lambda)(1-x)/[\alpha-\lambda+(1-\alpha)x] & \text{if} \quad x \in \left[0, 1-\dfrac{\lambda}{\alpha}\right] \\ \lambda & \text{if} \quad x \in \left[1-\dfrac{\lambda}{\alpha}, 1-\lambda\right] \\ 1-x & \text{if} \quad x \in [1-\lambda, 1] \end{cases} \tag{9b}$$

This map describes a periodic cycle with period two because its second iterate satisfies

$$h^2(x;\alpha) = h[h(x;\alpha);\alpha] = x \qquad \forall x \in [0,1].$$

Features of these cycles depend critically on the value of the collateral parameter $\lambda \in [0,1]$. If λ is sufficiently large, the demand for loans by borrowers is strong enough to drive the loan yield to its maximal value A. All capital is efficiently used by borrowers and equity pays the same as debt. The outcome is optimal with a high growth rate equal to βA, and the aggregate Solow residual equal to A. The economy grows smoothly along a balanced growth path with

$$c_t^i = (c_0^i)(\beta A)^t, \quad y_t = y_0(\beta A)^t, \quad R_t = A, \quad \forall t$$

To see this, we graph (9a) for $\lambda \in [1/2, 1]$ below.

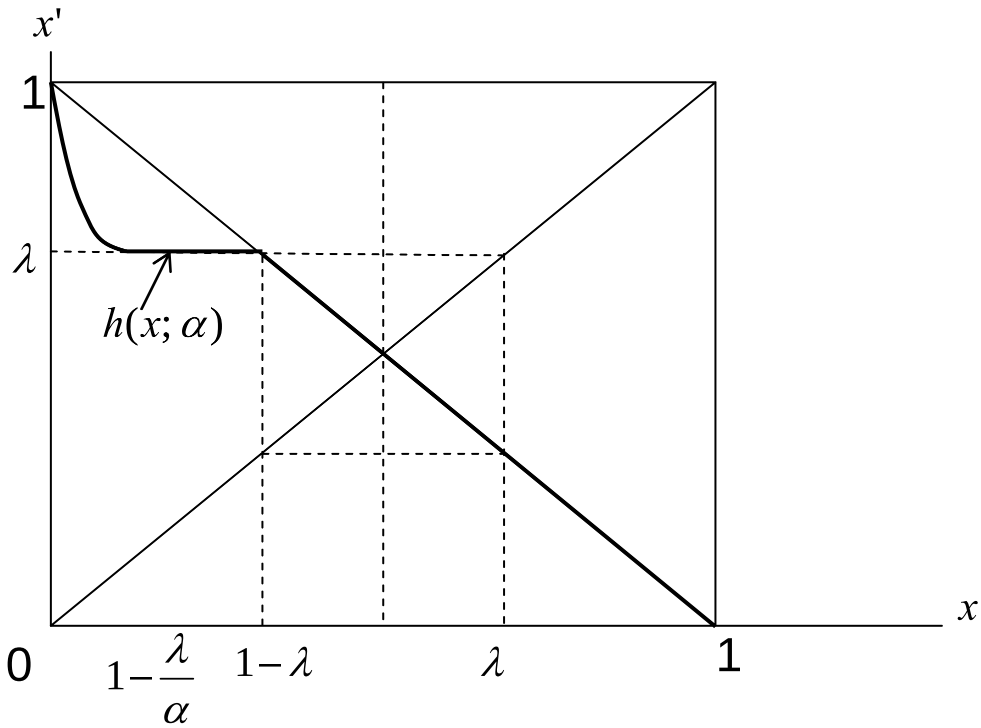


Figure 12(a). Deterministic cycles for $\lambda \in [1/2, 1]$

From this figure, it is clear that, starting from almost any $x_0 \in [0,1]$, the equilibrium sequence will converge in finitely many periods, to the cycle $x \in (x_1, x_2)$ with $x_1 + x_2 = 1$ and $x_2 \in [1/2, \lambda]$. Exceptions to this convergence are initial borrower wealth values at $x_0 = (0,1)$. In these two cases, the asymptotic equilibrium is the cycle $x = (0,1)$. However, the distribution of wealth is irrelevant for equilibrium outcomes when λ is large because all capital is employed by the best technology, and lenders enjoy the same yield as borrowers.

Suppose next that value of collateral is intermediate, i.e., $\lambda \in [\alpha/(1+\alpha), 1]$. Then a similar outcome occurs, the only difference being that equity yields more than debt and borrowers' consumption grows faster than lender's consumption. This situation is shown in Figure 12(b).

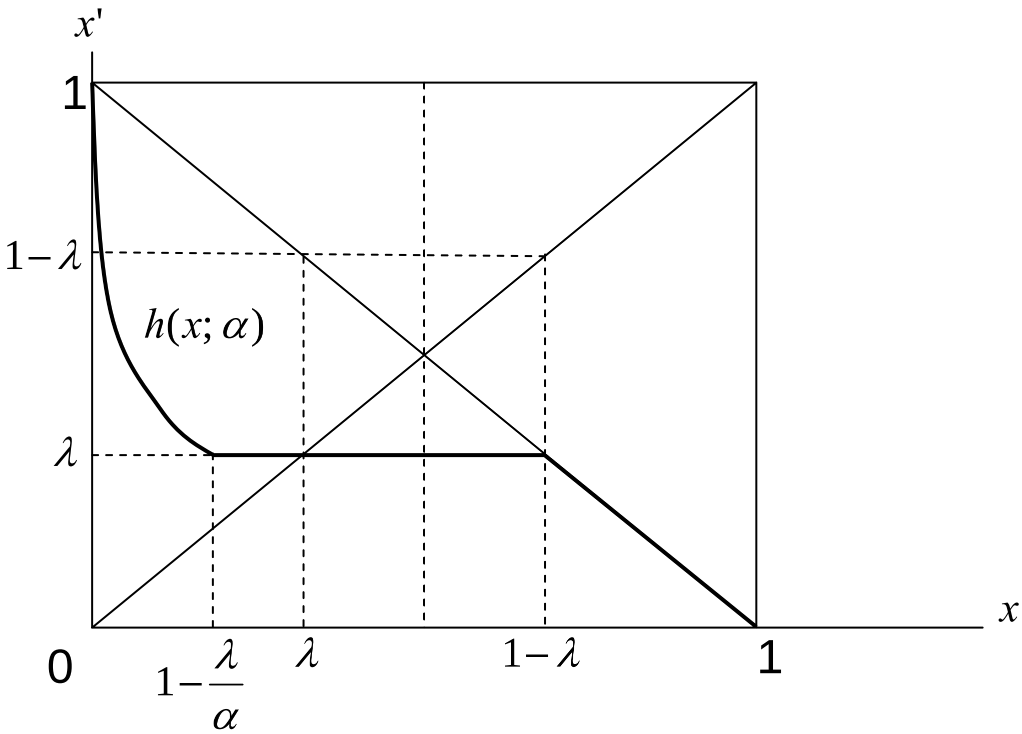


Figure 12(b). Deterministic cycles for $x \in \left[1-\frac{\lambda}{\alpha}, 1-\lambda\right]$

Equilibrium again converges to a two-cycle $(x_1, x_2) = (\lambda, 1-\lambda)$.

Lastly, we look at small collateral values $\lambda \in [0, \alpha/(1+\alpha)]$. The law of motion for this class of economies is in Figure 12(c).

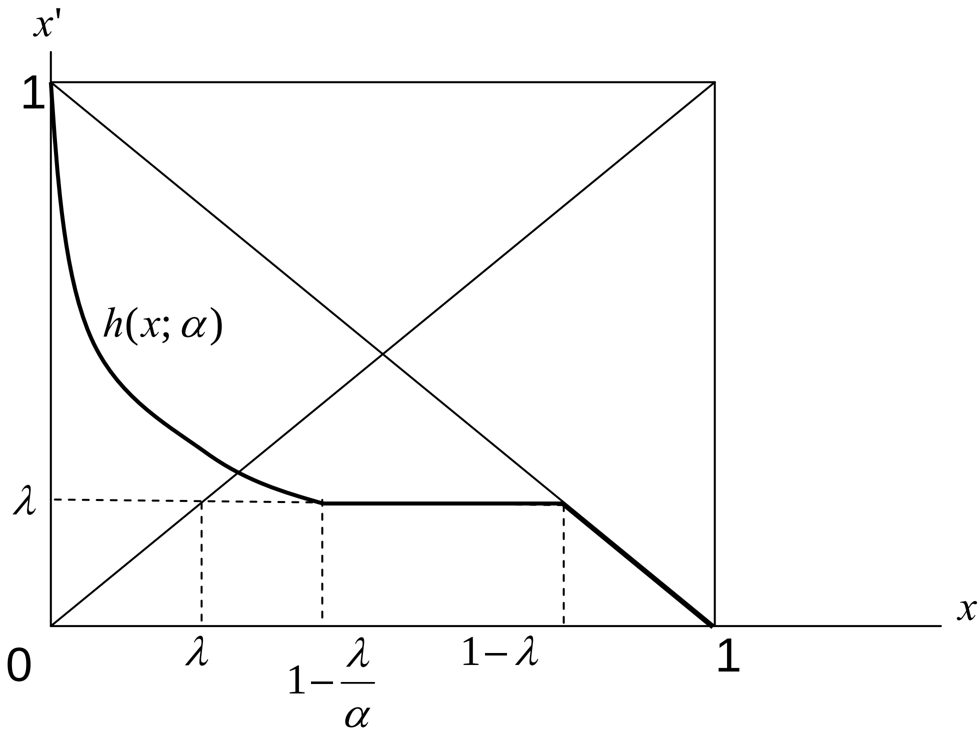


Figure 12(c). Deterministic cycles for $\lambda \in \left[0, 1-\frac{\lambda}{\alpha}\right]$.

Here we have an asymptotic invariant set $I = [\lambda, 1-\lambda]$ which attracts equilibrium sequences from any initial value $x_0 \in [0,1)$ of borrower's wealth. Clearly, for any $x_0 \in \left[1-\frac{\lambda}{\alpha}, 1-\lambda\right]$, the economy converges to the cycle $(1-\lambda, \lambda)$ in one period. Initial values $x_0 \in \left[1-\frac{\lambda}{\alpha}, 1-\lambda\right]$ converge to a cycle $\left(x_0, h(x_0;\alpha)\right)$ because the map h is periodic. In this environment with low collateral, financial markets do not do a very good job of directing capital to where it is most valued. Some production takes place in the less productive sector, and consumption growth rates are unequal for borrowers and lenders. Output growth fluctuates and is always below βA. We sum up our results in

<u>Proposition 7</u>. (a) Deterministic two-sector economies with pure collateral borrowing converge to a periodic cycle with period 2 which alternates between two values (x_1, x_2) for the borrower's equity share. For all values of $\lambda \in [0,1]$ we have

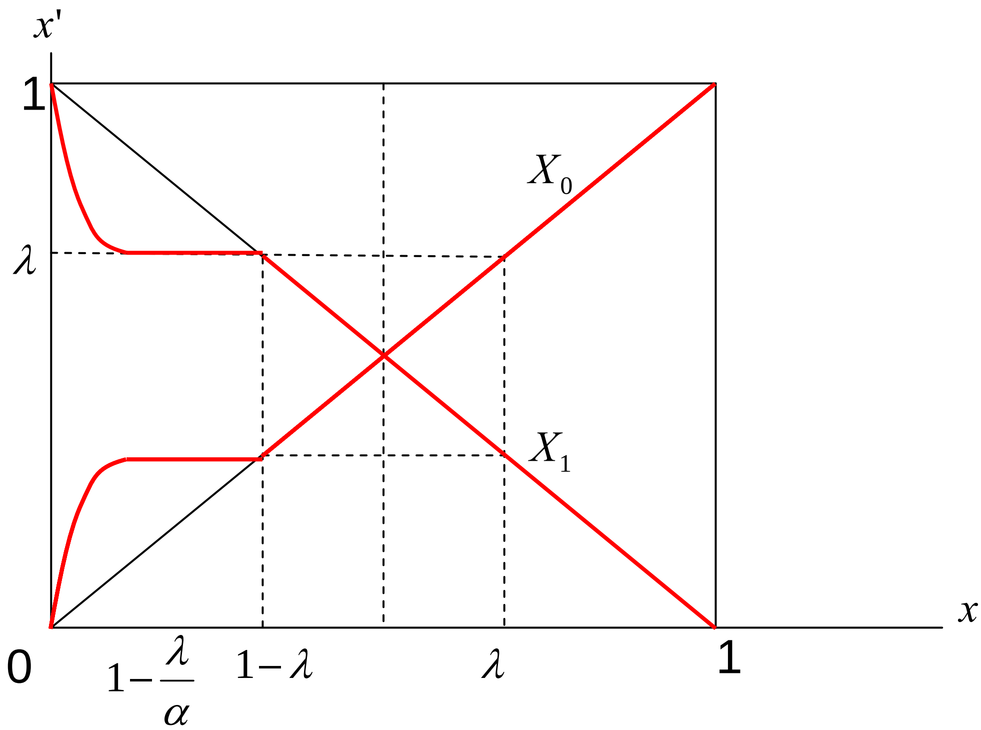
$$\min(\lambda, 1-\lambda) \le \min(x_1, x_2) \le \max(x_1, x_2) \le \max(\lambda, 1-\lambda)$$

(b) Both production and consumption achieve first-best values if $\lambda \in [1/2, 1]$; production is first best if $\lambda \in [\alpha/(1+\alpha), 1/2)$; both production and consumption are distorted if $\lambda \in [0, \alpha/(1+\alpha))$.

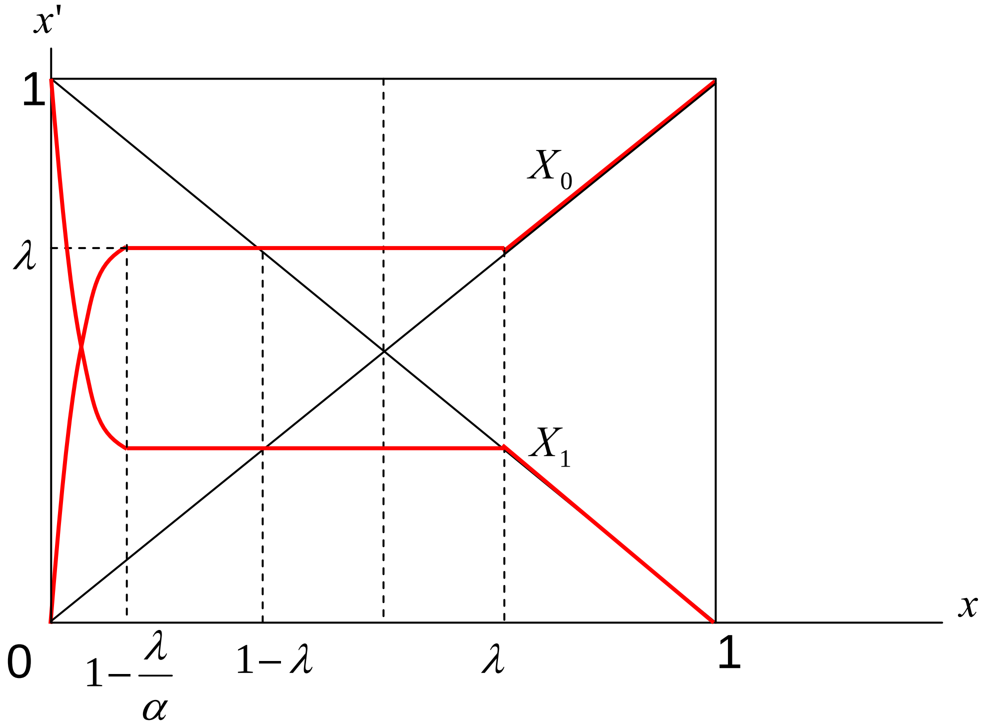
Problem 7.1: Suppose the deterministic economy described in this section consisting of $N \ge 2$ symmetric sectors whose productivities alternate between values $(\alpha A, A)$. Each sector has productivity A once every N periods, and αA the rest of the time. What values of the common collateral parameter λ will permit both aggregate production and aggregate consumption to attain their first best growth path?

4. Stochastic dynamics

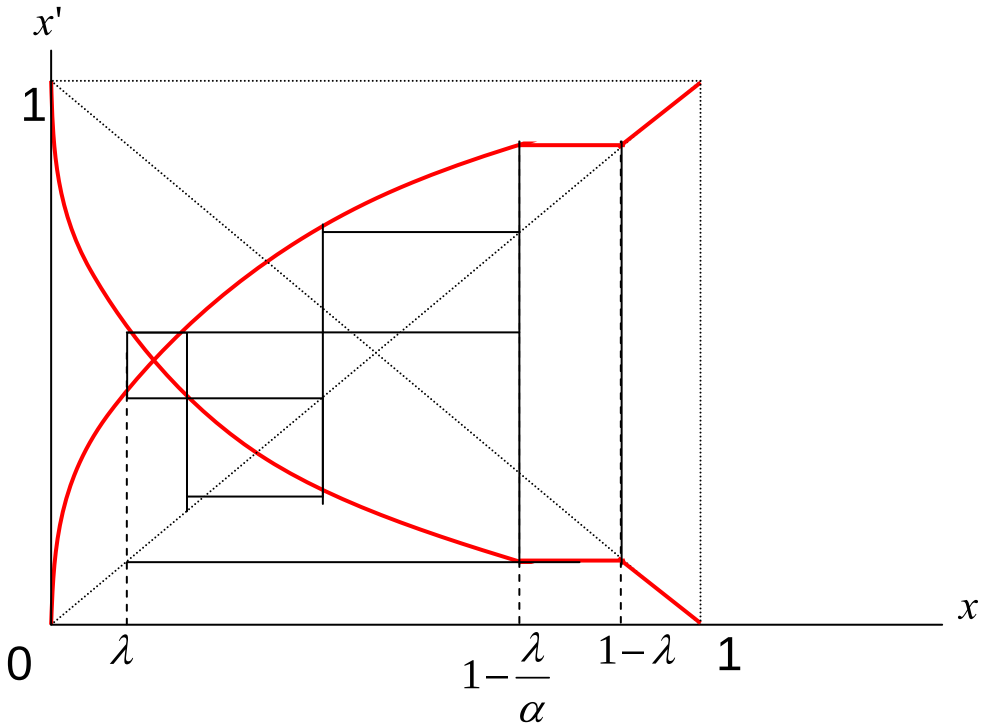
Stochastic equilibria are solutions to equation (8) which is graphed in Figure 13.



(a) $\lambda \ge 1/2$



(b) $1/2 > \lambda \geq \alpha/(1+\alpha)$



(c) $\lambda < \alpha/(1+\alpha)$

Figure 13.

The key result for these stochastic economies is that all production uses the efficient technology if there is enough collateral. The aggregate growth rate and the Solow residual are constant and high in this case. If the economy is short of collateral, then the borrower' equity share converges to a stochastic cycle with $2m+2$ states for some $m \geq 1$. Aggregate growth is βA in two of those states; it is slower in the remaining $2m+1$ states. Azariadis and Kaas (11/2008) prove the following general result.

Proposition: The dynamics of the borrower's wealth share converges to a finite stochastic cycle with probability one. In particular,

(a) Economies with large collateral, $\lambda \in [1/2, 1]$, converge to a cycle with two states (x_1, x_2) where $x_2 = 1 - x_1 \in [1-\lambda, \lambda]$. Production and consumption are efficient, debt constraints are slack, and aggregate growth equals βA.

(b) Economies with medium collateral, $\lambda \in [\alpha/(1+\alpha), 1/2)$ also converge to a two-state cycle $(x_1, x_2) = (\lambda, 1-\lambda)$. Production is efficient and aggregate growth is βA but sector growth rates are volatile, and borrowers are constrained for a fraction $1-\pi$ of periods.

(c) Economies with small collateral, $\lambda \in [0, \alpha/(1+\alpha))$ converge to a cycle with $2m+2$ states where m is the unique solution to the double inequality $f^{m-1}(\lambda;\alpha) < 1 - \lambda/\alpha \leq f^{m}(\lambda;\alpha)$ defined for the $(m-1)$-th and m-th iterates of the map f in equation (8).